Unified theory for localization physics in quasiperiodic lattices

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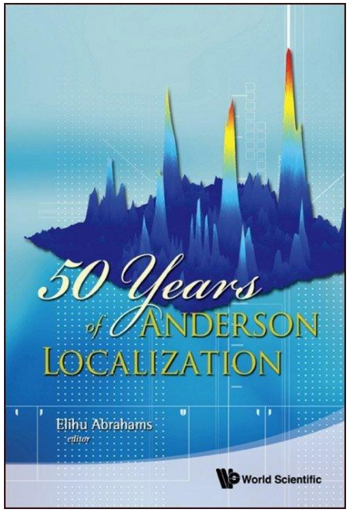
Anderson localization: background

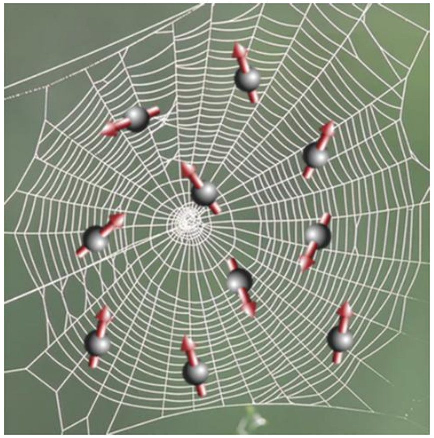
➢ **Random disorder**

Anderson localization (AL): disorder induces spatial localization of single-particle states

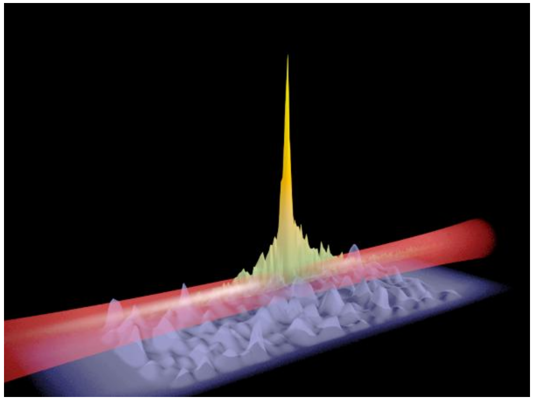
P. W. Anderson, Phys. Rev. 1958



Mechanism: Disorder drives the scattering between the forward and backward propagating modes, and leads to the localization of the wave functions depending on the dimensionality and disorder strength.



A very broad concept in modern physics



“Particles/spin are localized by spider web”

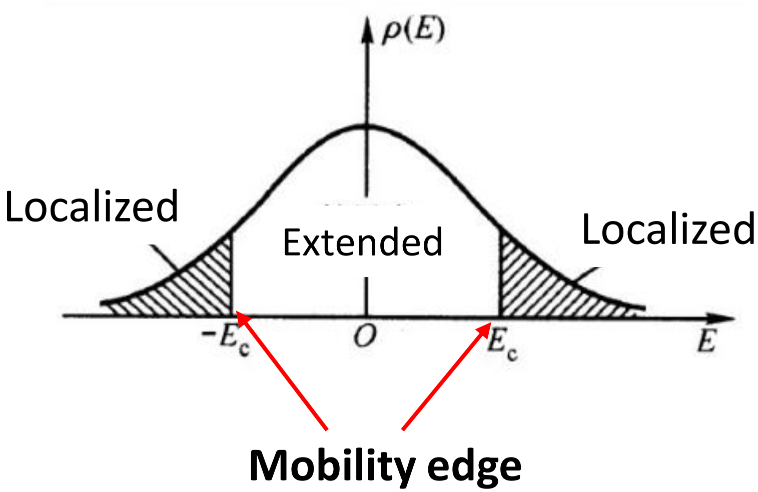
Scaling theory for Anderson localization [Gang of four (Phys. Rev. Lett. 42, 673, 1979)]

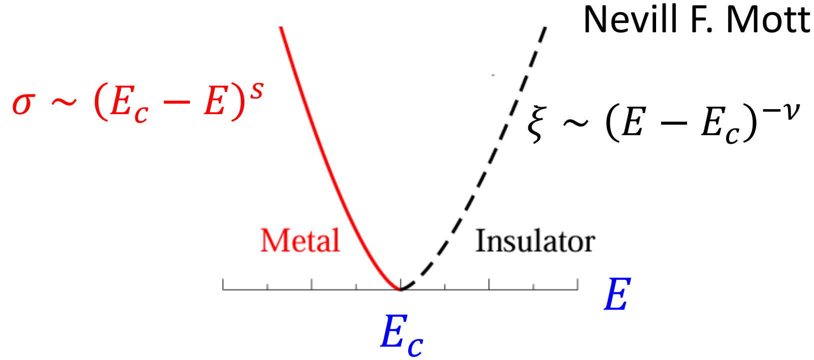
$$\beta(g) = \frac{d\ln(g)}{d\ln(L)} \qquad \lim_{G\to\infty} \beta(g) = d-2.$$

All states are localized for $d \leq 2$ (for spinless systems).

Mobility edges in 3D: coexisting localized-extended states in a single spectrum

Nevill F. Mott





Consequence: Metal-insulator transition by tuning the Fermi energy E_F across MEs $\pm E_c$

Universality classes of Anderson transition

Localization length diverges at Anderson transition point

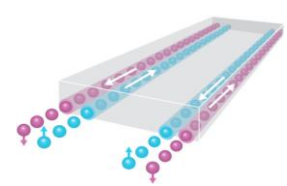
$$\xi \sim |W - W_c|^{-\nu}$$

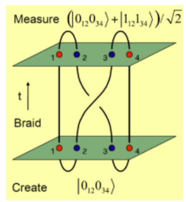
Tenfold way: time-reversal symmetry (TRS), particle hole symmetry (PHS), chiral symmetry

Class	TRS	PHS	Chiral	Family	Critical exponent ν
A	0	0	0	unitary	1.43 ± 0.04
AI	+1	0	0	orthogonal	1.57 ± 0.02
AII	−1	0	0	symplectic	1.35–1.39
AIII	0	0	1	chiral unitary	1.06 ± 0.02
BDI	+1	+1	1	chiral orthogonal	1.12 ± 0.06
CII	−1	−1	1	chiral symplectic	—
D	0	+1	0	BdG D	0.87 ± 0.03
C	0	−1	0	BdG C	0.996 ± 0.012
DIII	−1	+1	1	BdG DIII	0.96 ± 0.01
CI	+1	−1	1	BdG CI	1.17 ± 0.02

Standard approach: Nonlinear sigma model

A sophisticated & intrinsic connection to classification of topological phases





Ryu, Schnyder, Furusaki, Ludwig (2010)

Wegner (1979); Hikami (1981); Efetov (1983); Altland & Zirnbauer (1997)

Localization in quasiperiodic systems

- **1D Quasiperiodic (QP) lattice: the Aubry-Andre-Harper (AAH) model** [S. Aubry & G. André, Ann. Israel Phys. Soc 3, 18 (1980); Realization with cold atoms: Firenze-INFM-CNR group, *G. Roati et.al., Nature 453, 895 (2008)*]

$$H = J\sum_j (c_j^\dagger c_{j+1} + h.c.) + \sum_j \Delta\cos(2\pi\alpha j)\, c_j^\dagger c_j,$$

Incommensurate term

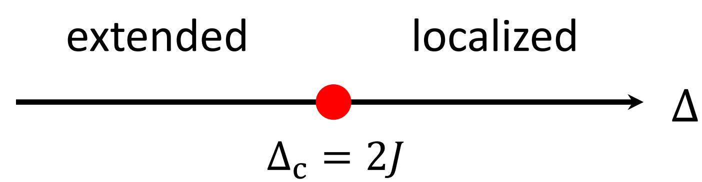
Self-dual point: the AAH Hamiltonian is invariant under the dual transformation at $\Delta = 2J$:

$$c_j = \frac{1}{\sqrt{L}}\sum_l e^{-i2\pi\alpha l x_j} b_l$$

local bases (c_j) — nonlocal bases (b_l)

$$H(c_j, c_j^\dagger) \longleftrightarrow H(b_j, b_j^\dagger)$$

Anderson transition: The critical point between extended ant localized phase:



At critical point: $\Delta_c = 2J$,
all eigenstates are critical

Mobility edges in QP systems: break the self-duality

- Extended-localized

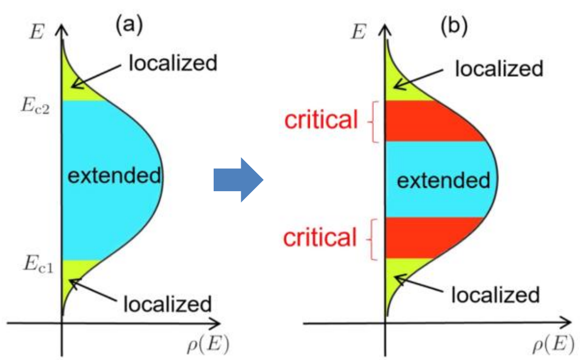
  Generalized AA model with long-range hopping:
  J. *Biddle & S. Das Sarma, PRL 104, 070601 (2010).*
  Generalized AA model with deformed onsite potential:
  S. Ganeshan, J. H. Pixley, S. Das Sarma, PRL 114, 146601 (2015).
  Type-I mosaic model with structured quasiperiodic onsite potential:
  Wang, Xia, Zhang, Yao, Chen, You, Zhou, XJL, PRL 125, 196604 (2020).

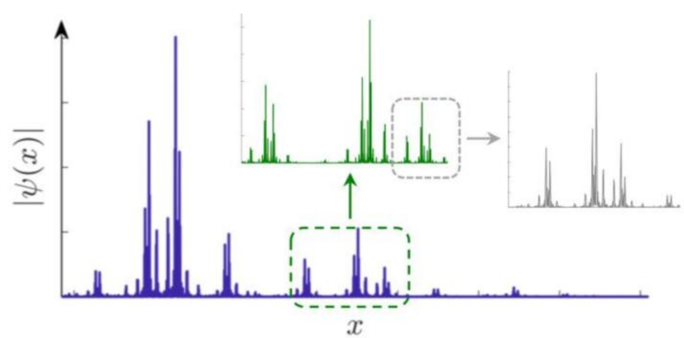
- Critical-localized

  Type-II mosaic model with structured quasiperiodic hopping potential:
  Zhou, Wang, Poon, Q. Zhou, XJL, PRL 131, 176401 (2023).

- *Extended-localized-critical ("tripartite phase")*

  Incommensurate AIII class topological insulator: *Y. Wang, L. Zhang, Sun, Poon, XJL, PRB 106, **L**140203 (2022).* See also: *Gonçalves, Amorim, Castro, & Ribeiro, PRL 131, 186303 (2023).*





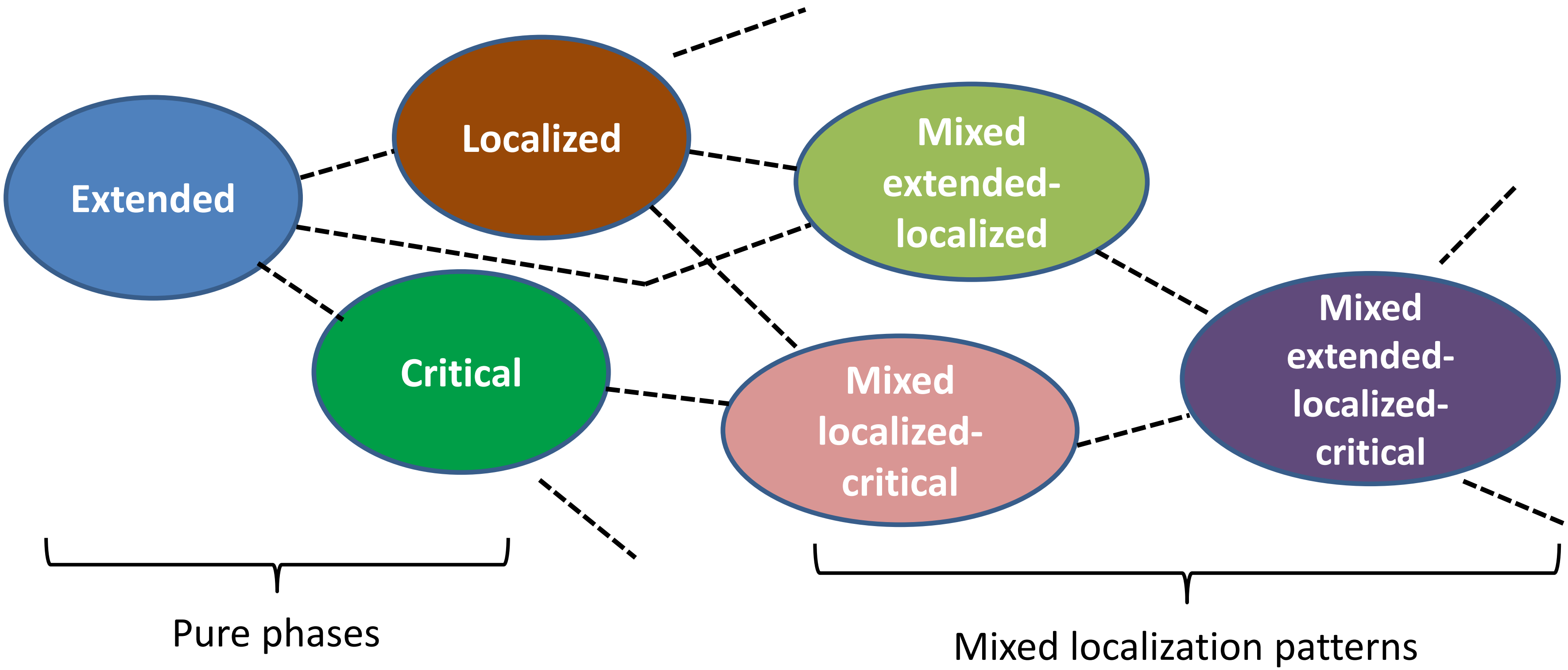
Multifractal critical states

Co-existence of extended-localized-critical states is exceptional for quasiperiodic systems.

XJL, Nat. Phys. 20, 1851 (2024).

Motivation

Beyond toy models, rich localization patterns can emerge in generic quasiperiodic systems:

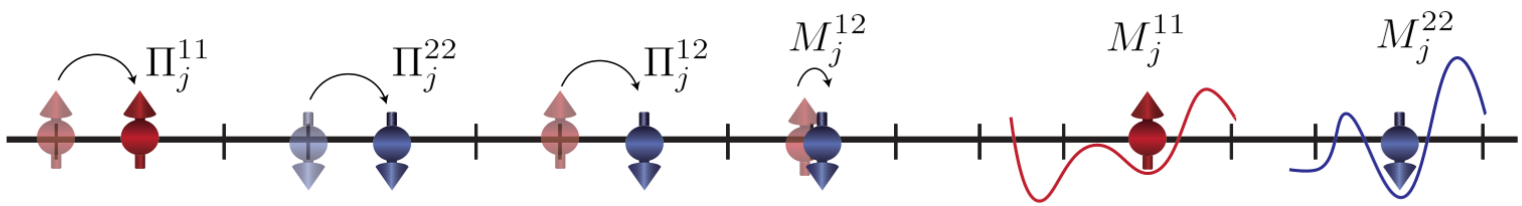


Question: any unified theory to characterize all of them? What are the universal mechanisms for the emergence of different type of localization patterns?

Unified framework of localization physics

The generic Hamiltonian describing the 1D spinful quasiperiodic (QP) lattice:

$$H = \sum_{j,s,s'} (c^{\dagger}_{j+1,s} \Pi_j^{s.s'} c_{j,s'} + \mathrm{h.c.}) + \sum_{j,s,s'} c^{\dagger}_{j,s} M_j^{s,s'} c_{j,s'}$$



Π_j: off-diagonal coupling matrix (with quasiperiodic hopping terms)

M_j: diagonal coupling matrix (with quasiperiodic on-site potentials)

Type I Mosaic model: $\Pi_j = t\sigma_+, M_j = t\sigma_x + 2\lambda V_j^{\mathrm{d}}$.

Type II Mosaic model: $\Pi_j = \lambda\sigma_+, M_j = 2tV_j^{\mathrm{d}}(\sigma_x + \sigma_0)$.

Incommensurate AIII TI model: $\Pi_j = t_0\sigma_z + it_{\mathrm{so}}\sigma_y, M_j = \eta m_z V_j^{\mathrm{d}}\sigma_z + (1-\eta)m_z V_j^{\mathrm{d}}\sigma_0$.

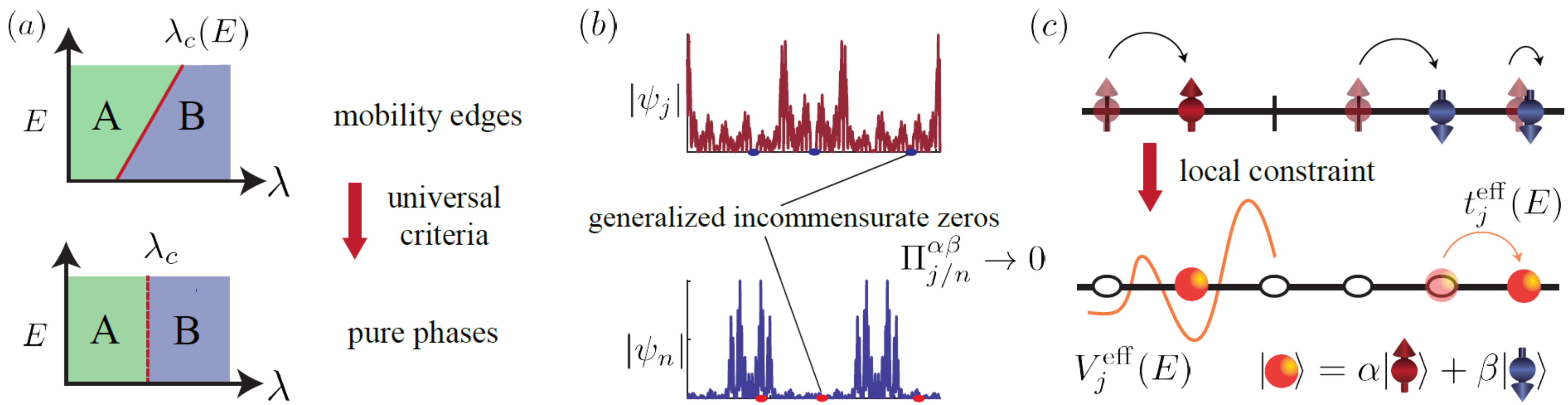
Generic 1D spinless chains: $\Pi_j = t_j\sigma_y, M_j = V_j\sigma_y$,

with Majorana representation: $c_j = \frac{1}{2}(\gamma_{A,j} + i\gamma_{B,j}), c_j^{\dagger} = \frac{1}{2}(\gamma_{A,j} - i\gamma_{B,j})$]

And beyond the known models, ……

X.-C. Zhou, B. -C. Yao, Y. Wang, Y. Wang, Y. Wei, Q. Zhou, XJL, *The fundamental localization phases in quasiperiodic lattices: a unified framework and exact results*, arXiv:2503.24380v5; Science Bulletin 71, 1654 (2026).

Main results



Universal result II: mechanism for emergence of critical states

Universal result I: Criteria for the pure phases without MEs

Universal result III: Exact solvability of QP systems from local constraint

Universal result I: Criteria for the pure phases without MEs

The system exhibits pure phase without mobility edges (MEs) if:

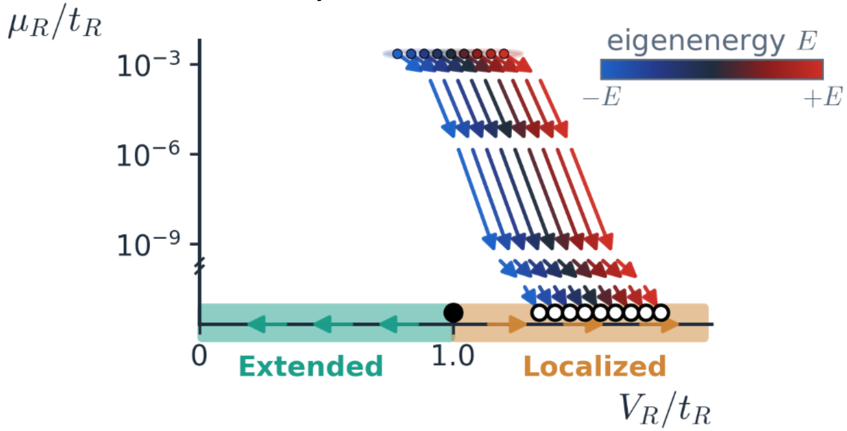
The on-site matrix M_j are purely quasiperiodic, and the hopping coupling matrix Π_j are either uniform or purely quasiperiodic, with the system satisfying the chiral(-like) symmetry

$$\sigma_y \Pi_j' \sigma_y^\dagger = -\Pi_j', \qquad \sigma_y M_j' \sigma_y^\dagger = -M_j'$$

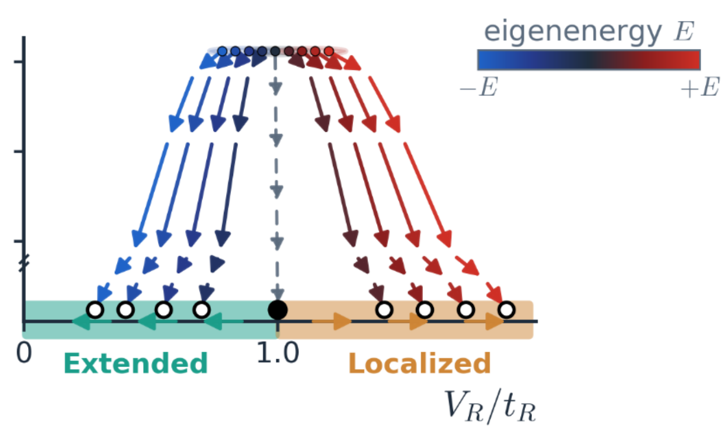
Properties: Chiral(-like) symmetry: the eigenstates come in pairs with symmetric energy $(E, -E)$

Renormalization group flow: at the fixed point: relevant parameters are energy-independent.

Pure phases without MEs



MEs: E_c



X.-C. Zhou, B. -C. Yao, Y. Wang, Y. Wang, Y. Wei, Q. Zhou, XJL, arXiv:2503.24380v5; Science Bulletin 71, 1654 (2026).

Universal result II: mechanism for emergence of critical states

Previous theory for spinless QP systems: Incommensurately distributed zeros (IDZs) in hopping.

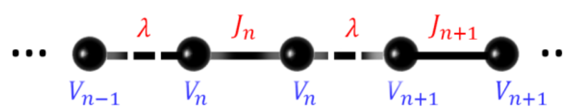
The type-II mosaic model in spinless represtation:

$$H = \sum_j (t_j a_j^\dagger a_{j+1} + \text{h.c.}) + \sum_j V_j n_j,$$

with quasiperiodic mosaic hopping structure:

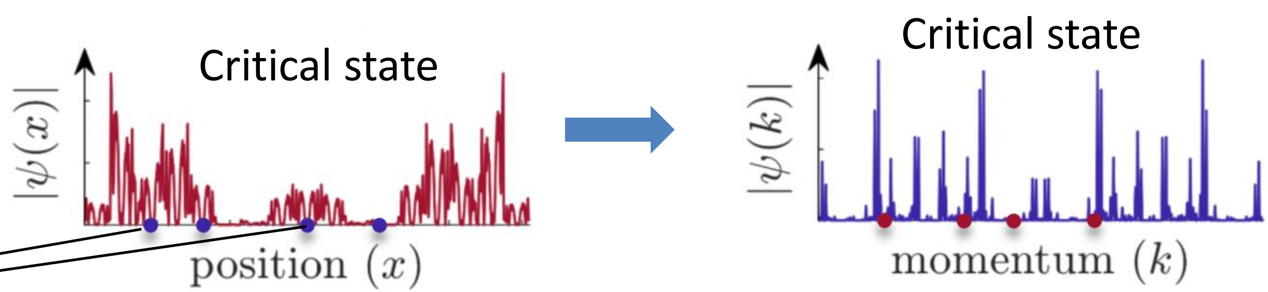
$$t_n = J_n \propto \cos(2\pi\alpha n + \theta)\text{ , for } n = 0 \bmod \kappa$$

For example, $\kappa = 2$



$J_n \propto \cos(2\pi\alpha n + \theta)$

Incommensurately distributed zeros:
$J_{n_\infty} \to 0$, for $L \to \infty$



X.-C. Zhou, Wang, Poon, Q. Zhou, XJL, PRL 131, 176401 (2023).

Universal result II: mechanism for emergence of critical states

Generalization to the present spinful quasiperiodic (QP) chains:

Spinful quasiperiodic systems can host critical states if the system possesses generalized incommensurate matrix element zeros $\mathcal{G}_{\Pi}$, which are defined as

$$\mathcal{G}_{\Pi} = \left\{ j_k \middle| \lim_{L\to\infty} \Pi^{\alpha\beta}_{j_k} = 0 \right\}$$

- Spinless: Incommensurately distributed zeros (IDZs) in hopping coefficients.
- Spinful: Generalized incommensurate zeros (GIZs) in hopping matrix elements.

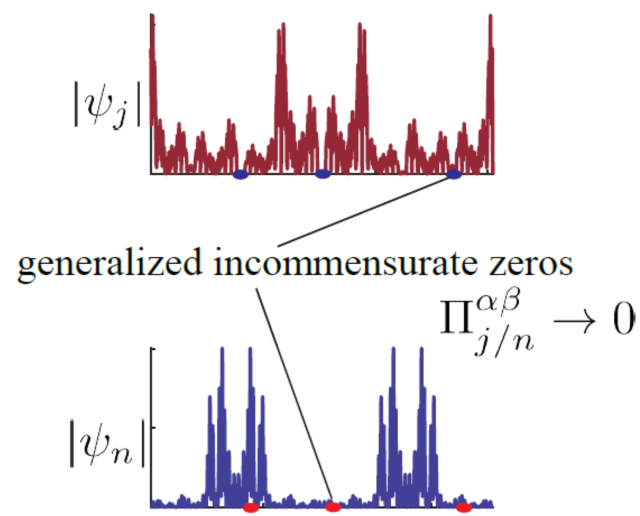
GIZs are invariant under spinful dual transformation:

$$V^{\mathrm{od}}_{j} c^{\dagger}_{j+1s} c_{js'} + \mathrm{h.c.}$$

$$\Updownarrow$$

$$V^{\mathrm{od}}_{n} c^{\dagger}_{n+1s} c_{ns'} + \mathrm{h.c.}$$

$\Rightarrow$



Introduce self-invariant critical orbitals

Universal result III: Exact solvability of QP systems from local constraint

The spinful QP chains are exactly solvable under following constrain:

A spinful quasiperiodic system can find exactly solvable points if the bond coupling matrix Π_j are degenerate, and coefficients do not contain QP modulations:

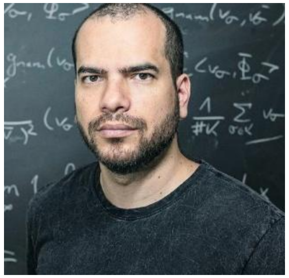
$$\det\left|\Pi_j\right| = 0, \qquad p_j^s = \text{Const.}$$

- Lyapunov exponent (inverse of localization length) can be computed from transfer matrix by Avila's global theory

$$\gamma_\epsilon(E) = \lim_{m\to\infty} \frac{1}{2\pi m} \int \ln \|\mathcal{T}_{m,1}(\theta + i\epsilon)\| d\theta,$$

which determines all the analytic properties of the states.

Quantum states	Energy spectrum
extended	absolutely continuous
Localized	point spectrum
Critical	singular continuous



Artur Avila, Acta. Math. 1, 215 (2015)
(A Field's medal work)

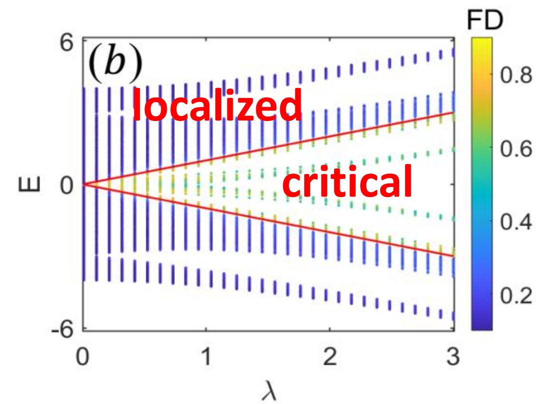
Y. Wang, Xia, Zhang, Yao, Chen, J. You, Q. Zhou, and XJL, PRL 125, 196604 (2020);
X.-C. Zhou, Wang, Poon, Zhou, XJL, PRL 131, 176401 (2023).

Application I: New models for critical states

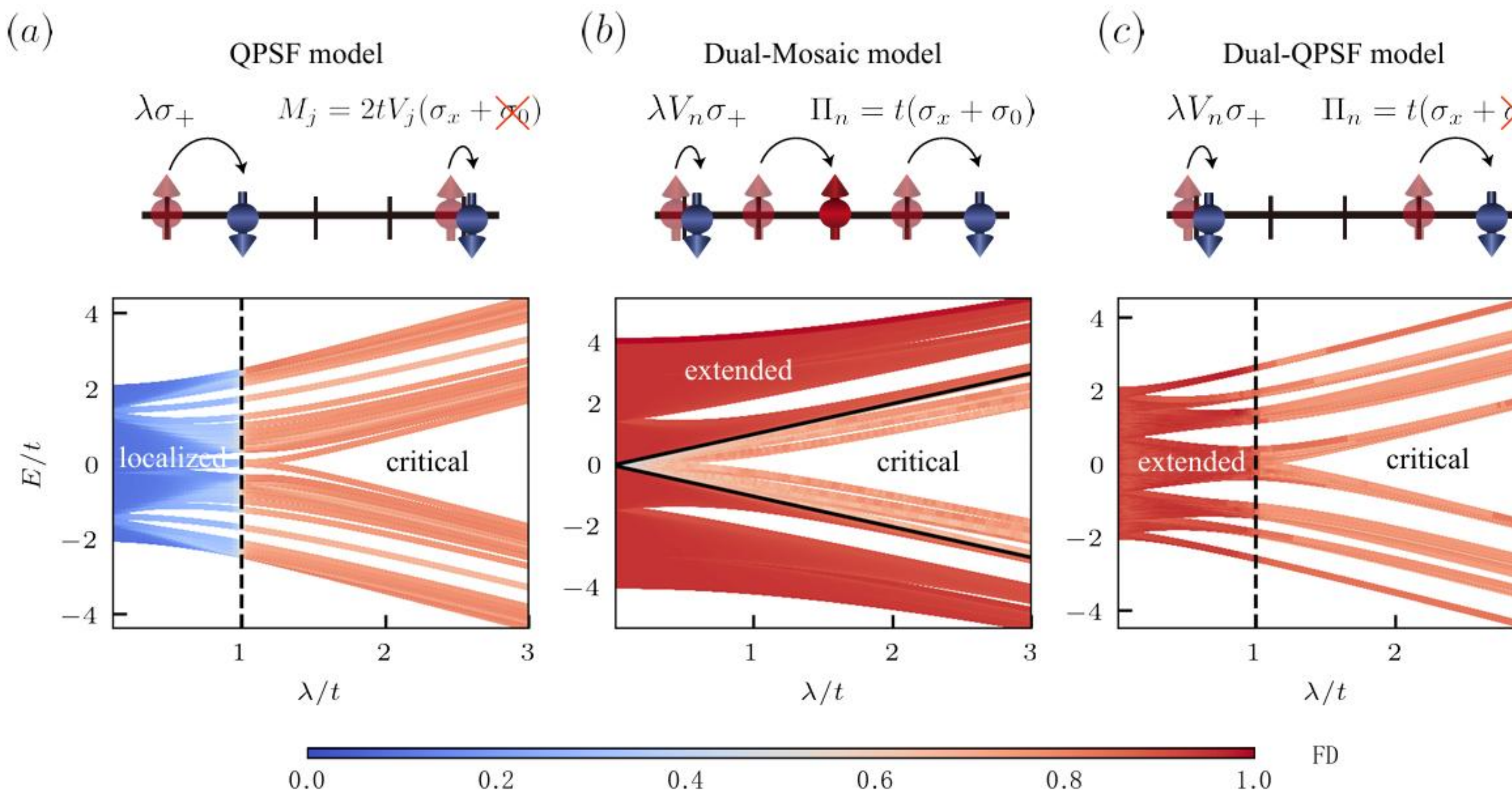
Start from type II QP mosaic model (mapped to spin-1/2 system):

$$H_{\mathrm{M-II}} = \sum_j \lambda (c_{j+1}^\dagger \sigma_- c_j + \mathrm{h.c.}) + 2t \sum_j V_j^{\mathrm{d}} c_j^\dagger (\sigma_0 + \sigma_x) c_j,$$

Break chiral symmetry



Construct new exactly solvable model:

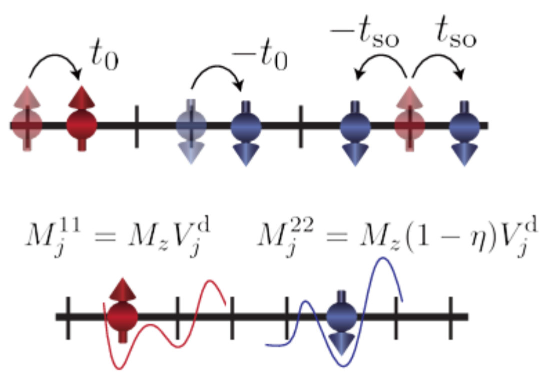


Application II: The model hosting the seven fundamental localization patterns

Quasiperiodic optical Raman lattice model:

$$\Pi_j = t_0\sigma_z + i t_{\rm so}\sigma_y$$

$$M_j = M_z V_j^{\rm d}[\eta\sigma_z + (1-\eta)\sigma_0]$$



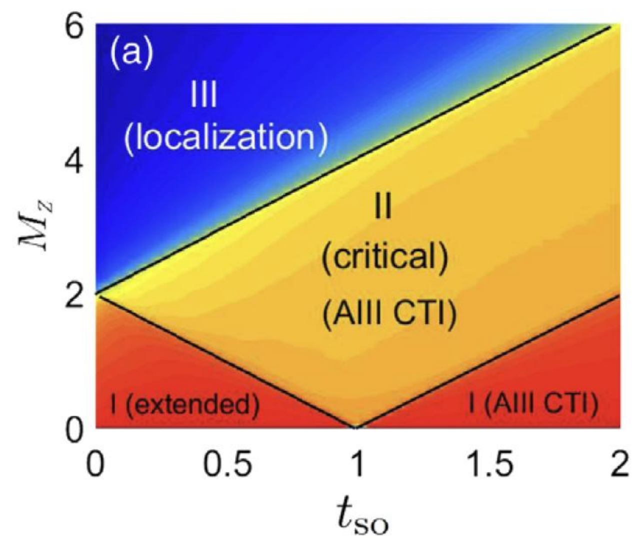
I) With Chiral symmetry at $\eta = 1$ (no σ_0 term)

$$\sigma_x(\Pi_j, M_j)\sigma_x^{-1} = -(\Pi_j, M_j)$$

Pure phases proved from RG theory, beyond Avila global theory.

Exact phase diagram

$(t_0 = 1)$



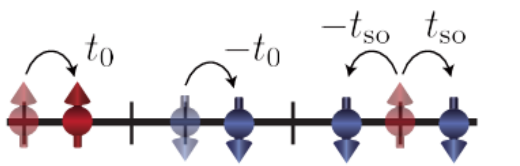
Y.-C. Wang, L. Zhang, S. Niu, D. Yu and XJL, PRL 125, 073204 (2020).

X.-C. Zhou, B. -C. Yao, Y. Wang, Y. Wang, Y. Wei, Q. Zhou, XJL, arXiv:2503.24380v5; Science Bulletin 71, 1654 (2026).

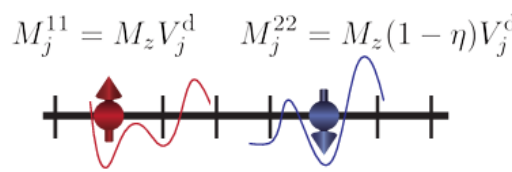
- Coexisting extended, localized, and critical states

II) For $\eta < 1$, spin-independent QP potential breaks chiral symmetry

$$\Pi_j = t_0\sigma_z + it_{\rm so}\sigma_y$$

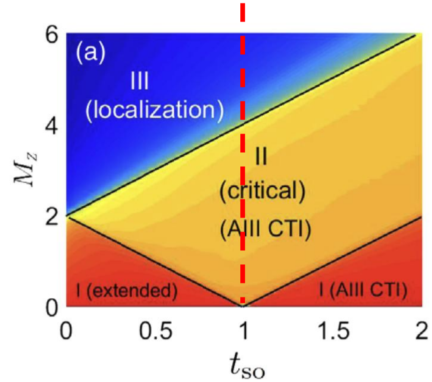


$$M_j = M_z V_j^{\rm d}[\eta\sigma_z + (1-\eta)\sigma_0]$$

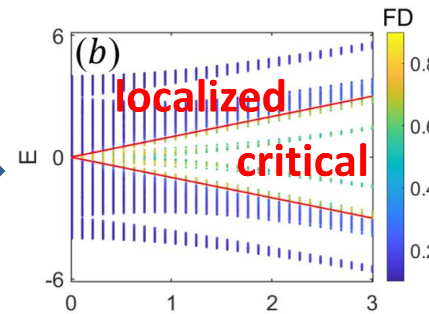


- Case I:

$t_{\rm so} = t_0$

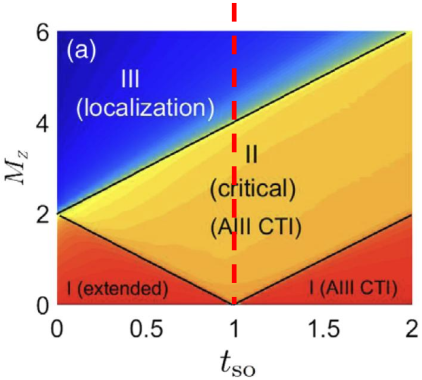


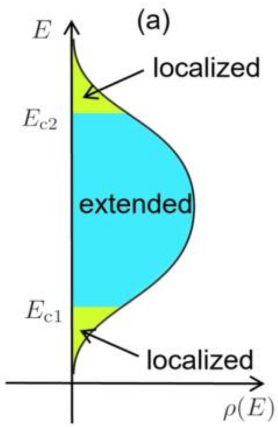
$\eta = 1/2$

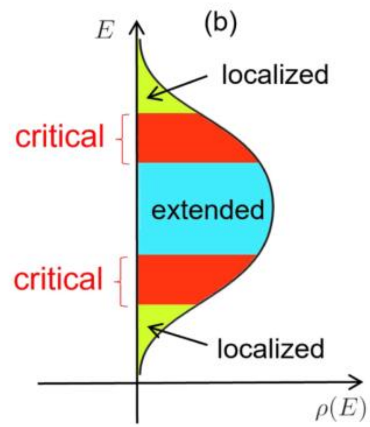


- Case II:

$t_{\rm so} < t_0$ $t_{\rm so} > t_0$



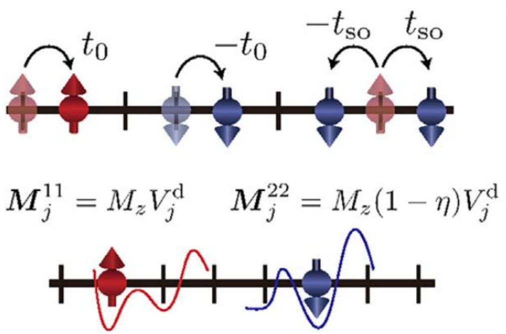


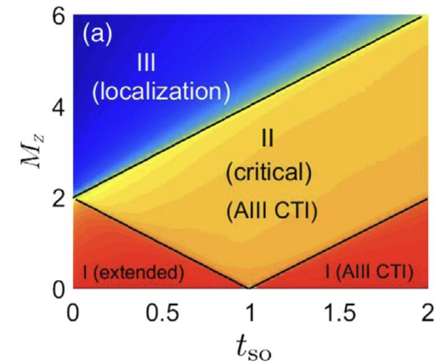


Tripartite phase with coexisting extended, localized, and critical states (Wang, Zhang, Sun, Poon, and XJL, PRB 106, L140203 (2022))

- Seven fundamental localization patterns in quasiperiodic systems

III) The generic full phase diagram





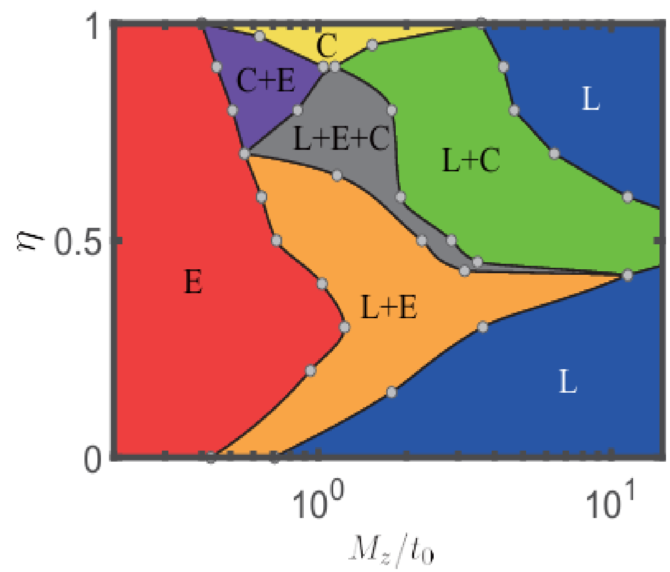
$\eta = 1$

Generic $\eta \leq 1$

Three pure phase:
Extended, localized, critical

Four mixed localization patterns:
Extended-localized,
Critical-localized,
Critical-extended,
Extended-critical-localized

$3 + 4 = 7$



L: localized

E: extended

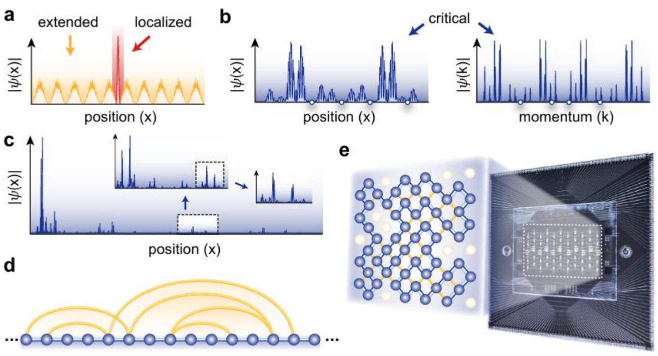
C: critical

Phase diagram for $t_{so} = 0.5t_0$

X.-C. Zhou, B. -C. Yao, Y. Wang, Y. Wang, Y. Wei, Q. Zhou, XJL, *The fundamental localization phases in quasiperiodic lattices: a unified framework and exact results*, arXiv:2503.24380v5; Science Bulletin 71, 1654 (2026).

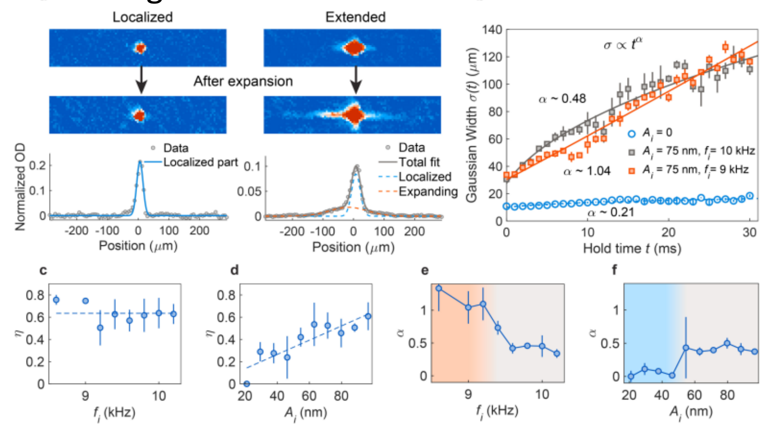
Experimental results

- Superconducting qubit platform: observation of exact critical states



W. Huang, X.-C. Zhou, ..., Y. Zhong*, XJL*, and D. Yu*, arXiv:2502.19185, Nature Physics (online) (2026).

- Optical lattice： Observation of tripartite phase for coexisting extended-localized-critical states



Z.-S. Hu, Y. -J. Guo, Y.-D. Wei, ..., S. -J. Jin*, and XJL*, arXiv:2605.21441.

- Type-II QP mosaic spin model with long-range couplings

$$H = \sum_j J_j(\sigma_j^\dagger \sigma_{j+1}^- + \sigma_{j+1}^\dagger \sigma_j^-) + \sum_j J_{\rm nn}(\sigma_j^\dagger \sigma_{j+2}^- + \sigma_{j+2}^\dagger \sigma_j^-),$$

Analytic threshold of transition from critical to extended states via RG theory (beyond Avila theory).

(I) With only next nearest-neighboring coupling:

$$J_{\rm nn} > \max\left(J, \sqrt{J\lambda}\right),$$

(II) With next-next-nearest-neighboring coupling:

$$J_{\rm nn} > \sqrt{J \max\left(J, \lambda, \mu\right) - \lambda\mu}.$$

Summary

1. Established a unified and rigorous framework for all the fundamental localization states in the quasiperiodic systems.

2. Exactly solvable models are proposed for all fundamental localization patterns, with some being observed in experiment.

References:

X.-C. Zhou, B. -C. Yao, Y. Wang, Y. Wang, Y. Wei, Q. Zhou, and XJL, *Science Bulletin* 71, 1654 (2026).
W. Huang, X.-C. Zhou, …, Y. Zhong, XJL, and D. Yu, *arXiv: 2502.19185*; *Nature Physics (Published Online)* (2026).
Z.-S. Hu, Y. -J. Guo, Y.-D. Wei, B.-C. Yao, Z. Qian, X.-C. Zhou, B.-Z. Wang, J. Yang, X. Chen, S. -J. Jin, XJL, *arXiv: 2605.21441*.

See also previous references:

PRL 125, 196604 (2020); PRL 125, 073204 (2020). PRB 106, **L**140203 (2022); PRL 131, 176401 (2023).

Many future issues for the rigorous theory:

Large spin systems;
High dimensions;
Classification of critical states;
Open quantum systems;
Many-body interacting effects (MBL, many-body critical phases, etc).

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Theory

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Prof. Long Zhang (PKU→Huazhong Univ. of Sci & Tec)

Experiment

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Yajing Guo
Dr. Yu-Dong Wei
Jianing Yang
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Thanks for your attention!