Entanglement Enhanced Sensing and Metrology

A quantum information perspective

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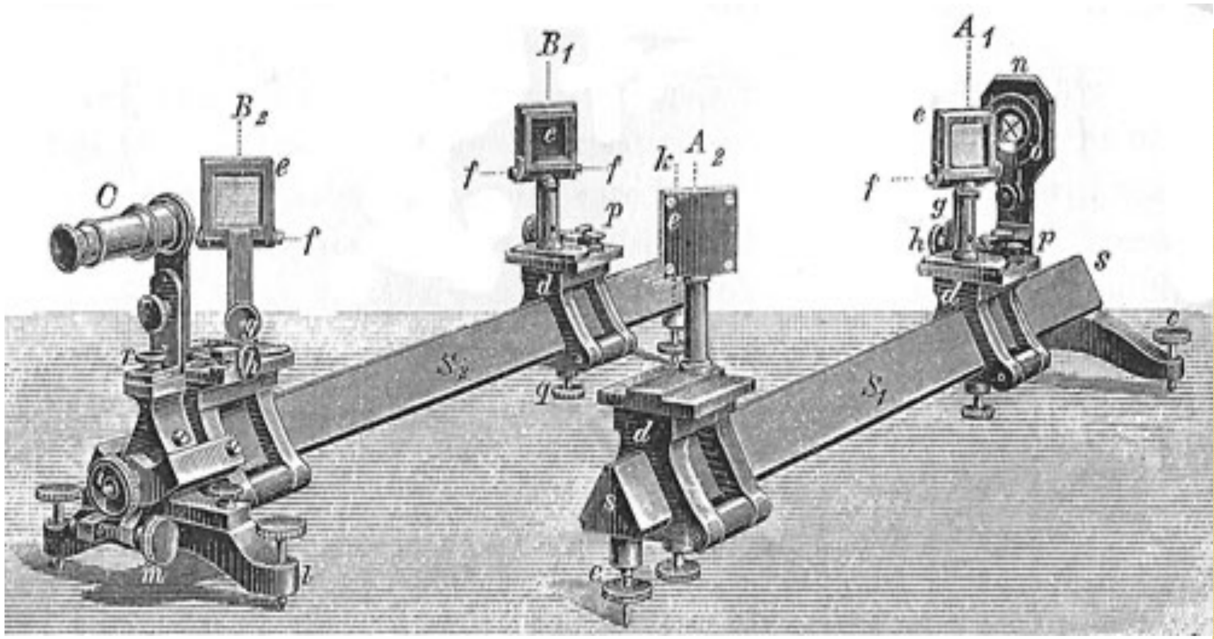
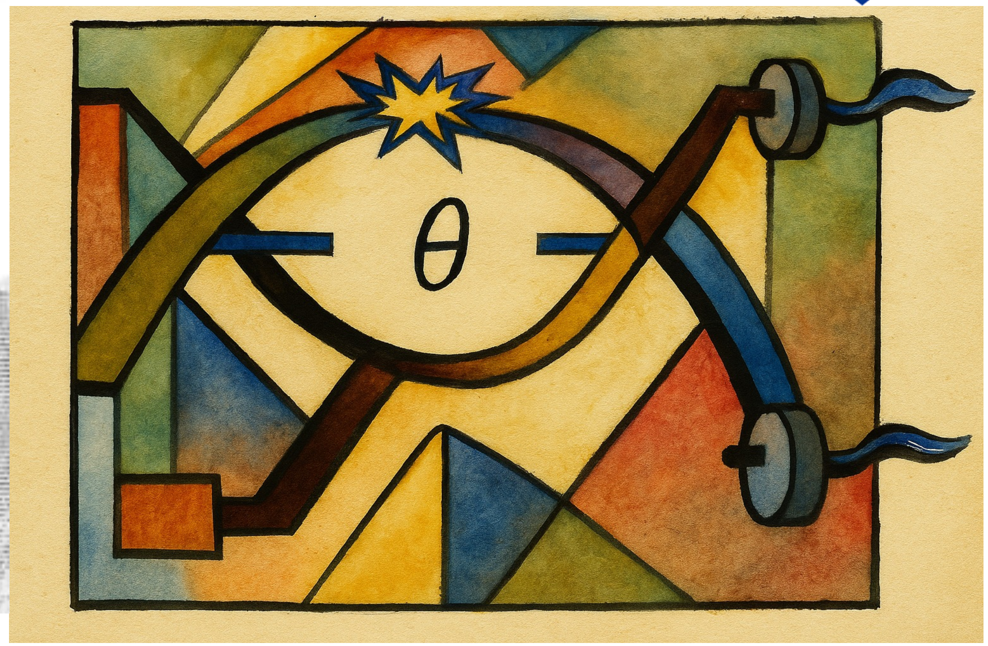




Outline

Entanglement

What is the physical difference between separability and entanglement?
Useful entanglement?

Multipartite Entanglement

i) How can we detect multipartite entanglement?
ii) How can we characterize multipartite entanglement?

Quantum Theory of Sensing and Metrology

i) Mach-Zehnder interferometer
ii) Theory of phase estimation: Frequentist approach
iii) Entanglement as a resource for phase estimation
iv) Theory of phase estimation: Bayesian approach

SEPARABILITY VS. ENTANGLEMENT

PRODUCT STATES

A

Sicily, silver Dekadrachm
405BC to 380BC

B

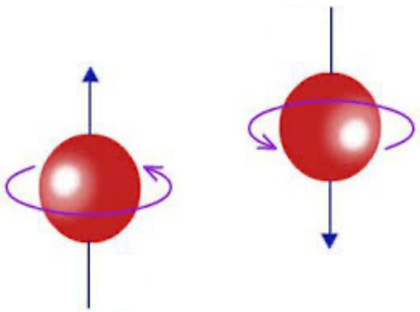
Classical bits

$$P(a,b) = \delta_{(a,tail)}\delta_{(b,head)}$$ pure state

$$P(a,b) = P_A(a)\ P_B(b)$$

product random state

E.g.: By independently tossing two coins or spins we get a product state.



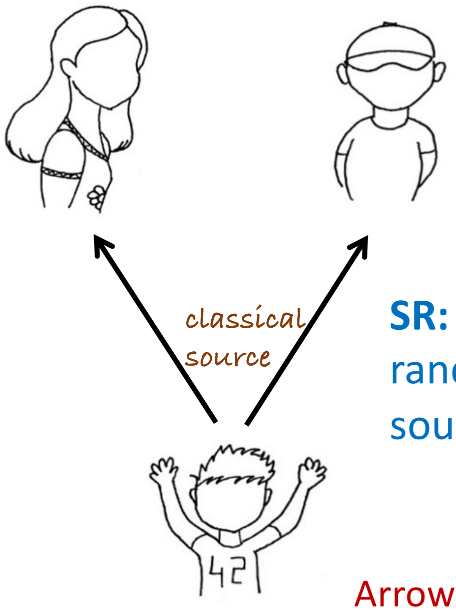
Qubits

$$|\psi\rangle = |\uparrow\rangle_A |\downarrow\rangle_B$$ pure state

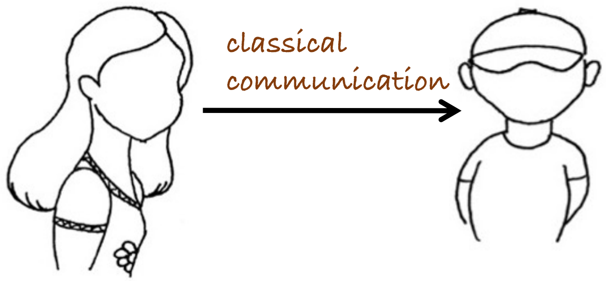
$$\hat{\rho}_{AB} = \hat{\rho}_A \otimes \hat{\rho}_B$$

product density matrix

SEPARABLE STATES FROM SHARED RANDOMNESS (SR) AND/OR LOCAL OPERATIONS AND CLASSICAL COMMUNICATION (LOCC)



SR: Alice and Bob receive correlated random records from a common source and then act locally.



LOCC: Alice and Bob can perform arbitrary local quantum operations on their own systems, and they can exchange classical messages.

Arrows indicate classical causal links: sharing or exchange of classical records λ

$$\hat{\rho}_{AB} = \sum_{\lambda} P(\lambda)\ |\psi_A^{\lambda}\rangle\langle\psi_A^{\lambda}| \otimes |\psi_B^{\lambda}\rangle\langle\psi_B^{\lambda}|$$

Shared randomness and LOCC are foundational concepts in quantum information because they define the limits of classical and local quantum processing,
(cannot create entanglement, can still decrease entanglement)
and serve as **benchmarks against which we understand quantum advantage**.

Most current quantum technologies use separable resources.

Quantum advantage requires special correlations that go beyond those created using only shared randomness and LOCC.

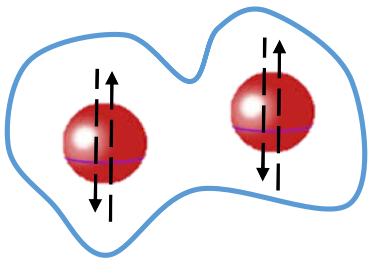
Entanglement: What is it?

ENTANGLED STATES: NON-SEPARABLE STATES

Quantum mechanics predicts states shared by Alice and Bob that cannot be written as any convex mixture of product states, and cannot be created from initially unentangled systems using only shared randomness and LOCC.

$$\hat{\rho}_{AB} \neq \sum_{\lambda} P(\lambda)\ |\psi_A^\lambda\rangle\langle\psi_A^\lambda| \otimes |\psi_B^\lambda\rangle\langle\psi_B^\lambda|$$

Example:



$$\frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\uparrow\rangle$$

Entangled states contains coherent superposition of product states.

How entanglement is different from classical correlations?

Example:

pure entangled state	vs.	separable mixture
$\vert\psi\rangle = \frac{1}{\sqrt{2}}\vert\uparrow\uparrow\rangle + \frac{1}{\sqrt{2}}\vert\downarrow\downarrow\rangle$		$\hat{\rho} = \frac{1}{2}\vert\uparrow\uparrow\rangle\langle\uparrow\uparrow\vert + \frac{1}{2}\vert\downarrow\downarrow\rangle\langle\downarrow\downarrow\vert$

$$\hat{\rho}_{AB} = |\psi\rangle\langle\psi| = \frac{1}{2}|\downarrow\downarrow\rangle\langle\downarrow\downarrow| + \frac{1}{2}|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + \frac{1}{2}|\downarrow\downarrow\rangle\langle\uparrow\uparrow| + \frac{1}{2}|\uparrow\uparrow\rangle\langle\downarrow\downarrow|$$

pure entangled state

"classical correlations"
(identical to separable mixtures)

contribution from quantum coherences
(superposition of product states)

$$P(a,b) = P(a,b)_{cc} + C(a,b)_{qc}$$

Entangled states provide correlations that can be stronger than those provided by separable states

Quantum coherences can provide an extra burst

This extra contribution is widely believed to be the necessary resource for quantum advantage.

Remember:
For a generic observable

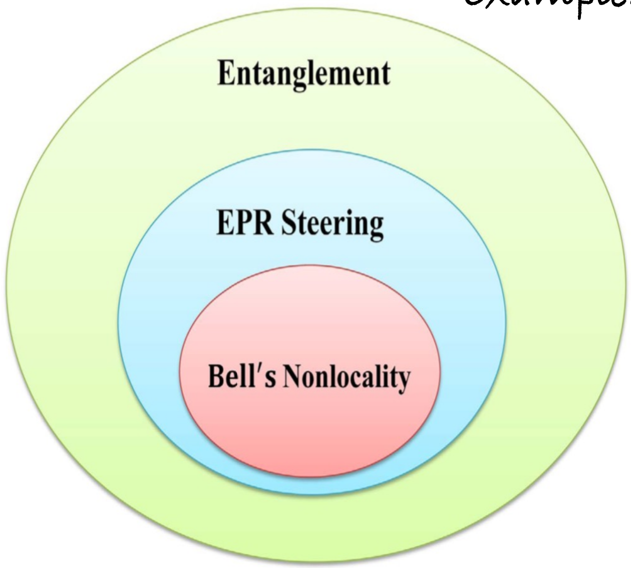
$$P(x) = Tr[\hat{\rho}\ \hat{E}_x]$$

THE PATH FROM ENTANGLEMENT TO USEFUL RESOURCES

i) Certify multipartite entanglement

$$\hat{\rho}_{A,B,C,\ldots} \neq \sum_{\gamma} P(\gamma)\, |\psi_A^\gamma\rangle\langle\psi_A^\gamma| \otimes |\psi_B^\gamma\rangle\langle\psi_B^\gamma| \otimes |\psi_C^\gamma\rangle\langle\psi_C^\gamma| \ldots$$

Example:



ii) Identify task-useful quantum correlations

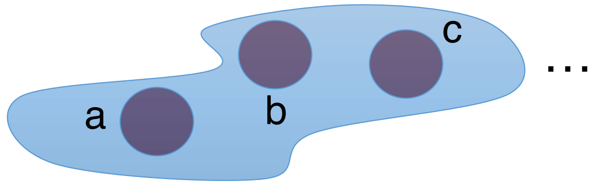
Entanglement is necessary for many quantum advantages, but it is not sufficient: only task-useful correlations matter

Question: which entanglement is useful for sensing and metrology?

HOW TO RECOGNIZE ENTANGLEMENT?

MULTIPARTITE ENTANGLEMENT CERTIFICATION IS A HARD TASK

unscrambling classical and quantum correlations is exponentially costly



$$\hat{\rho} = \sum_{\gamma} p_{\gamma} \, |\psi^{\gamma}_{a,b,c\ldots}\rangle\langle\psi^{\gamma}_{a,b,c\ldots}| \quad = \quad \sum_{j} q_{j} \, |\phi^{j}_{a,b,c\ldots}\rangle\langle\phi^{j}_{a,b,c\ldots}|$$

source of classical correlations among states of the ensemble

source of quantum correlations among the parties in each state of the ensemble

the state admits different decompositions (purely quantum feature)

Classical and quantum correlations in the ensemble are, to some extent, interchangeable

To recognize multipartite entanglement it is necessary to scrutinize all possible decompositions of the state

Homework:

A classical mixture of entangled states is not necessarily entangled

Simple example:

$$|\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle \pm |0,1\rangle)$$

$$|\phi_\pm\rangle = \frac{1}{\sqrt{2}}(|1,1\rangle \pm |0,0\rangle)$$

Consider the four Bell's states: quantum entangled states that violate Bell's locality

The balanced statistical mixture of "maximally" entangled Bell states is "maximally" separable.

In other words: The quantum correlations from each Bell state cancel out in the density matrix.

$$\hat{\rho} = \frac{1}{4}[|\psi_+\rangle\langle\psi_+| + |\psi_-\rangle\langle\psi_-| + |\phi_+\rangle\langle\phi_+| + |\phi_-\rangle\langle\phi_-|] = \frac{1}{4}\hat{I}$$

proportional to Identity: Mixed state with no correlations.

A state is entangled if it cannot be written as a convex combination of separable pure states in any possible decomposition

$$\hat{\rho} \neq \sum_\gamma p(\gamma)\ |\psi_1^\gamma\rangle\langle\psi_1^\gamma| \otimes |\psi_2^\gamma\rangle\langle\psi_2^\gamma|$$

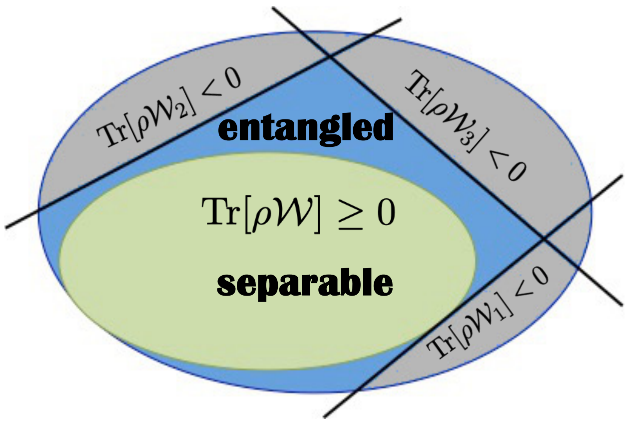
We need a sufficient criteria that
i) certify and characterize multipartite entanglement that is
ii) experimentally accessible and iii) can be efficiently calculated.

Typical criteria are "algebraic": look for entropic functions or observables that can only take specific set of values when calculated with separable states.

E.g.: Entanglement witnesses detect all entangled states by using different observables W with

$$\mathrm{Tr}[\rho\mathcal{W}] < 0 \qquad \mathrm{Tr}[\rho_{\mathrm{sep}}\mathcal{W}] \geq 0$$

detect different entangled states by
using different witness observables W.



Horodecki et al PLA 1997

Entanglement as speed in state space

$\hat{\rho}$ ⟷ compare $e^{-i\hat{H}\tau}\ \hat{\rho}\ e^{i\hat{H}\tau}$

Is this state entangled?

Central idea: entangled states can evolve faster than separable states

L. Pezzè and A. Smerzi, PRL 2009

"Statistical distances" quantify distinguishability between states.

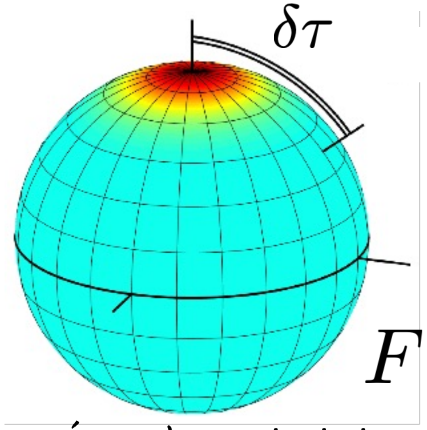
"Statistical speeds" quantify the infinitesimal rate at which distinguishability changes along an evolution.

$$S(\tau) = \frac{d}{d\tau} D(\tau)$$

"speed" and "distance" in the Hilbert space.

Goal: characterize "dynamically" multipartite entanglement

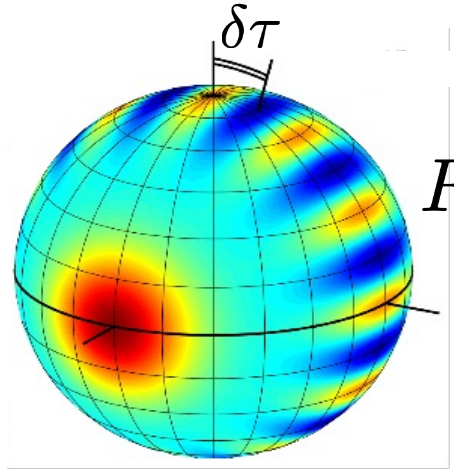
Back-of-the-envelope example:



$F \sim N$

spin coherent state

$|\psi\rangle = |N, 0\rangle$



$F \sim N^2$

Noon state

$|\psi\rangle = \frac{1}{\sqrt{2}}(|N, 0\rangle + |0, N\rangle)$

Fubini-Study distance and speed of *nearby* states:

$$d_{FS}(\tau) \equiv \arccos|\langle\psi(\tau)|\psi\rangle| \simeq \frac{1}{2}\sqrt{F}\,\tau + O(\tau^2)$$

$$v_{\mathrm{FS}} \equiv \frac{d\ d_{\mathrm{FS}}}{d\tau} = \frac{\sqrt{F}}{2}$$

Fisher information $F(\hat{\rho}, \hat{H})$ sets the local speed of distinguishability

Distance between probability distributions

Introduce a function that, in analogy with distances among points in space, measures how far away (how much different) are two probability distributions:

i) $D[p(x), q(x)] \geq 0$

ii) $D[p(x), q(x)] = D[q(x), p(x)]$

iii) $D[p(x), q(x)] = 0 \; iff \; p(x) = q(x)$

iv) $D[p(x), q(x)] + D[q(x), r(x)] \geq D[p(x), r(x)]$

$triangular \; inequality$

Consider a probability distribution that depends continuously on a parameter

and speed $S(\theta) = \lim_{\epsilon \to 0} \frac{D[p(x|\theta + \epsilon), p(x|\theta)]}{\epsilon}$

Hellinger distance and its speed of evolution of two neighboring states

$$D_H^2 = \frac{1}{2}\sum_x (\sqrt{p_x(\tau)} - \sqrt{p_x})^2)$$

Hellinger distance

$$S_H^2 \equiv F = \sum_x p_x(\theta)\left(\frac{\partial}{\partial\theta}\log p_x(\theta)\right)^2$$

Fisher information

$$P(x|\theta) = Tr[\hat{\rho}(\theta)\hat{E}_x]$$

Homework: calculate the distance between two Gaussian distributions

$$P_1(x) = \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-(x-\bar{x}_1)^2/2\sigma_1^2}$$

$$P_2(x) = \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-(x-\bar{x}_2)^2/2\sigma_2^2}$$

$$d_H^2 = 1 - \int_{-\infty}^{+\infty} dx \sqrt{P_1(x)} \sqrt{P_2(x)} =$$

$$= \sqrt{\frac{2\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2}} \; e^{-\frac{(\bar{x}_1-\bar{x}_2)^2}{\sigma_1^2+\sigma_2^2}}$$

Homework: calculate the distance between two Poissonian distributions

$$p_1 = e^{-\lambda_1} \lambda_1^n / n! \qquad p_2 = e^{-\lambda_2} \lambda_2^n / n_2!$$

$$d_H^2 = 1 - e^{-1/2 \left(\sqrt{\lambda_1} - \sqrt{\lambda_2}\right)^2}$$

Homework: distinguishability of two events

Are more distinguishable the events with probability distribution

1. $\{0.5, 0.5\}$ and $\{0.55, 0.45\}$

or

2. $\{1, 0\}$ and $\{0.95, 0.05\}$

Homework: distinguishability of two events

Are more distinguishable the events with probability distribution

① $\{0.5, 0.5\}$ and $\{0.55, 0.45\}$

or

② $\{1, 0\}$ and $\{0.95, 0.05\}$

① We have a coin and want to know if it is biased or not.

With a single toss, it is impossible to decide.

We need to toss the coin many times.

② We have an event that should occur with certainty. However, rare negative events might happen. How many events, in average, I should look at to be confident that rare negative events do not occur?

① $d_H = \frac{1}{2} \sum_x \left(\sqrt{p(x)} - \sqrt{q(x)}\right)^2 =$

$$= \frac{1}{2}\left[\left(\sqrt{0.5} - \sqrt{0.55}\right)^2 + \left(\sqrt{0.5} - \sqrt{0.45}\right)^2\right]^{1/2} = 0.025$$

② $d_H = \frac{1}{2}\left[\left(\sqrt{1} - \sqrt{0.95}\right)^2 + \left(\sqrt{0} - \sqrt{0.05}\right)^2\right]^{1/2} = 0.11$

Homework:

Fisher Information from Hellinger Distance

Let $p(x|\theta)$ be a smooth parametric family of probability distributions. The (squared) Hellinger distance between $p(x|\theta)$ and $p(x|\theta + \delta\theta)$ is:

$$d_H(\theta, \theta + \delta\theta) = \sqrt{1 - \sum_x \sqrt{p(x|\theta)\, p(x|\theta + \delta\theta)}}$$

Step 1: Taylor expansion

Expand $p(x|\theta + \delta\theta)$ to second order:

$$p(x + \delta\theta) = p(x) + \delta\theta\, \partial_\theta p(x) + \frac{1}{2}\delta\theta^2\, \partial_\theta^2 p(x) + \mathcal{O}(\delta\theta^3)$$

Step 2: Expand the square root

$$\sqrt{p(x)p(x + \delta\theta)} = p(x)\left[1 + \frac{\delta\theta}{2p(x)}\partial_\theta p(x) - \frac{\delta\theta^2}{8p(x)^2}(\partial_\theta p(x))^2 + \cdots\right]$$

Step 3: Sum over x

$$\sum_x \sqrt{p(x)p(x+\delta\theta)} = \sum_x p(x) + \frac{\delta\theta}{2}\sum_x \partial_\theta p(x) - \frac{\delta\theta^2}{8}\sum_x \frac{(\partial_\theta p(x))^2}{p(x)} + \cdots$$

$$= 1 - \frac{\delta\theta^2}{8}\sum_x \frac{(\partial_\theta p(x))^2}{p(x)} + \mathcal{O}(\delta\theta^3)$$

since $\sum_x p(x) = 1$ and $\sum_x \partial_\theta p(x) = 0$.

Step 4: Plug into d_H

$$d_H(\theta, \theta+\delta\theta) = \sqrt{1 - \left(1 - \frac{\delta\theta^2}{8}\mathcal{I}(\theta) + \cdots\right)} = \sqrt{\frac{\delta\theta^2}{8}\mathcal{I}(\theta) + \cdots}$$

$$= \delta\theta \cdot \sqrt{\frac{1}{8}\mathcal{I}(\theta)} + \mathcal{O}(\delta\theta^2)$$

Result

$$\boxed{d_H(\theta, \theta+\delta\theta) = \delta\theta \cdot \sqrt{\frac{1}{8}\sum_x \frac{(\partial_\theta p(x|\theta))^2}{p(x|\theta)}} + \mathcal{O}(\delta\theta^2)}$$

Thus, Fisher information $\mathcal{I}(\theta)$ is the square of the local statistical speed in the Hellinger geometry.

The value of the Fisher information of a quantum state depends on our choice of the observable:

POVM

$$\begin{cases} \sum_x \hat{E}_x = \hat{I} \\ \langle \hat{E}_x \rangle \geq 0 \end{cases}$$

$$P_x(\theta) = Tr[\hat{\rho}(\theta)\hat{E}_x]$$

$$F = \sum_x p_x(\theta) \left(\frac{\partial}{\partial \theta} \log p_x(\theta) \right)^2$$

the same quantum state
can give different classical data

$$\hat{\rho}$$

$$p_a(x) = Tr[\hat{\rho}\ \hat{E}_x^a]$$

$$p_b(x) = Tr[\hat{\rho}\ \hat{E}_x^b]$$

$$\cdots$$

$$p_l(x) = Tr[\hat{\rho}\ \hat{E}_x^l]$$

Question:
Which POVM maximizes the Fisher information of a quantum states?
Optimize over all measurements.

From Classical to Quantum Fisher information

$$\left(\frac{1}{2}\sum_x(\sqrt{p_x}-\sqrt{q_x})^2\right)^{1/2} \xrightarrow[\text{POVM}]{\text{max}} \left(1-Tr\left[\sqrt{\sqrt{\hat\rho}\;\hat\sigma\sqrt{\hat\rho}}\right]\right)^{1/2}$$

Hellinger distance — Bures-Helstrom distance

$$\Big\downarrow \frac{\partial}{\partial\theta} \qquad\qquad \Big\downarrow \frac{\partial}{\partial\theta}$$

$$F_{cl} = \sum_x p_x(\theta)\; L_x(\theta)^2 \xrightarrow[\text{POVM}]{\text{max}} F_Q = Tr[\hat\rho\;\hat L^2]$$

classical Fisher information — quantum Fisher information

$$= \left(1-|\langle\psi(\theta|\psi\rangle|\right)^{\frac{1}{2}}$$

with pure states

$$= 4\;\Delta^2\hat H$$

$$\frac{\partial P_x(\theta)}{\partial\theta} = L_x(\theta)\;P_x(\theta) \qquad\qquad \frac{\partial\hat\rho(\theta)}{\partial\theta} = \frac{1}{2}\{\hat L,\hat\rho\}$$

Notice: i) $F_Q \geq F_{cl}$ ii) quantum Fisher independent of POVM

Helstrom 1976
Wooters 1982
Braunstein & Caves, 1994

This note explain the eigenbasis formula for the quantum Fisher information of a mixed state undergoing a unitary parameter encoding. The formula is

$$F_Q(\rho, \hat{H}) = 2 \sum_{k,l\,:\,p_k+p_l>0} \frac{(p_k - p_l)^2}{p_k + p_l} \left|\langle k|\hat{H}|l\rangle\right|^2 . \tag{1}$$

Here

$$\rho = \sum_k p_k |k\rangle\langle k| \tag{2}$$

is the spectral decomposition of the density matrix, with $p_k \geq 0$ and $\sum_k p_k = 1$, and $\hat{H}$ is the Hermitian generator of the parameter shift. Equation (1) shows explicitly how mixedness reduces the quantum statistical speed with respect to the pure-state result $F_Q = 4\Delta^2 \hat{H}$.

Unitary parameter encoding

We consider a parameter τ encoded as

$$\rho_\tau = e^{-i\tau\hat{H}}\rho\, e^{i\tau\hat{H}}. \tag{3}$$

The eigenvalues of ρ_τ are the same as those of ρ. The parameter does not change the spectrum; it rotates the eigenvectors. The derivative at $\tau = 0$ is

$$\partial_\tau \rho_\tau|_{\tau=0} = -i[\hat{H}, \rho]. \tag{4}$$

In the eigenbasis of ρ, this gives

$$\langle k|\partial_\tau \rho_\tau|l\rangle_{\tau=0} = -i(p_l - p_k)\langle k|\hat{H}|l\rangle. \tag{5}$$

Thus, only the off-diagonal matrix elements of $\hat{H}$ in the eigenbasis of ρ can change the state. Diagonal elements of $\hat{H}$ merely generate phases inside the same eigenvector and do not modify the density matrix.

Symmetric logarithmic derivative

The quantum Fisher information is defined through the symmetric logarithmic derivative L_τ, which is the Hermitian operator satisfying

$$\partial_\tau \rho_\tau = \frac{1}{2}\left(\rho_\tau L_\tau + L_\tau \rho_\tau\right). \tag{6}$$

The quantum Fisher information is

$$F_Q(\rho_\tau) = \mathrm{Tr}(\rho_\tau L_\tau^2). \tag{7}$$

At $\tau = 0$, in the eigenbasis of ρ, Eq. (6) gives

$$\langle k|\partial_\tau \rho_\tau|l\rangle = \frac{p_k + p_l}{2}\langle k|L|l\rangle. \tag{8}$$

Therefore, whenever $p_k + p_l > 0$,

$$\langle k|L|l\rangle = \frac{2\langle k|\partial_\tau \rho_\tau|l\rangle}{p_k + p_l}. \tag{9}$$

If $p_k + p_l = 0$, the corresponding matrix element lies entirely outside the support of ρ and does not contribute to the quantum Fisher information.

Derivation of the eigenbasis formula

Using Eq. (7), we write

$$F_Q = \sum_k p_k \langle k|L^2|k\rangle = \sum_{k,l} p_k \left|L_{lk}\right|^2, \tag{10}$$

where $L_{lk} = \langle l|L|k\rangle$. Substituting Eq. (9) and symmetrizing the ordered pair (k, l) gives

$$F_Q = 2 \sum_{k,l\,:\,p_k+p_l>0} \frac{\left|\langle k|\partial_\tau \rho_\tau|l\rangle\right|^2}{p_k + p_l}. \tag{11}$$

Finally, inserting Eq. (5) yields

$$F_Q(\rho, \hat{H}) = 2 \sum_{k,l\,:\,p_k+p_l>0} \frac{(p_k - p_l)^2}{p_k + p_l} \left|\langle k|\hat{H}|l\rangle\right|^2. \tag{12}$$

Equivalently, since the terms (k, l) and (l, k) are equal, one can write

$$F_Q(\rho, \hat{H}) = 4 \sum_{k<l\,:\,p_k+p_l>0} \frac{(p_k - p_l)^2}{p_k + p_l} \left|\langle k|\hat{H}|l\rangle\right|^2. \tag{13}$$

Physical interpretation

The factor

$$\frac{(p_k - p_l)^2}{p_k + p_l} \tag{14}$$

has a simple meaning. A unitary generated by $\hat{H}$ can make the state distinguishable only by mixing eigenvectors of ρ that have different populations. If $p_k = p_l$, the corresponding contribution vanishes. Thus, rotations inside equally populated subspaces do not change the density matrix in a statistically visible way.

The formula also shows why mixed states are generally slower than pure states. For a pure state, all weight is concentrated on one eigenvector, so coherences between that eigenvector and the orthogonal subspace contribute maximally. For a mixed state, the contrast between eigenvalues is reduced, and the useful quantum speed is suppressed.

Pure-state limit

For mixed states, one always has

$$F_Q(\rho, \hat{H}) \leq 4\Delta_\rho^2 \hat{H}, \tag{15}$$

where

$$\Delta_\rho^2 \hat{H} = \mathrm{Tr}(\rho \hat{H}^2) - \mathrm{Tr}(\rho \hat{H})^2. \tag{16}$$

To see this from Eq. (12), note that

$$\frac{(p_k - p_l)^2}{p_k + p_l} \leq p_k + p_l. \tag{17}$$

Thus

$$F_Q(\rho, \hat{H}) \leq 2 \sum_{k,l} (p_k + p_l) |\langle k|\hat{H}|l\rangle|^2 = 4\,\mathrm{Tr}(\rho \hat{H}^2). \tag{18}$$

The quantum Fisher information is invariant under the shift $\hat{H} \mapsto \hat{H} - c\mathbb{I}$, because the identity commutes with ρ. Choosing $c = \mathrm{Tr}(\rho \hat{H})$ gives Eq. (15).

Equality holds for pure states. For mixed states the inequality is generally strict, because part of the variance of $\hat{H}$ can be classical fluctuation rather than useful quantum distinguishability.

Fisher information of *N* qubits in a ***separable*** state

$$\hat{\rho}_{cl} = \sum_{\gamma} P_{\gamma}\ \hat{\rho}_1^{\gamma} \otimes ... \hat{\rho}_N^{\gamma} \qquad \text{and} \qquad \hat{H} = \sum_{i=1}^{N} \hat{\sigma}_{\vec{n}_i}$$

$$F[\hat{\rho}_{cl}, \hat{H}] \leq \sum_{\gamma} P_{\gamma}\ F[\hat{\rho}_1^{\gamma} \otimes ... \hat{\rho}_N^{\gamma}, \hat{H}] = \sum_{\gamma} P_{\gamma} \sum_{i=1}^{N} F(\hat{\rho}_i^{\gamma}, \hat{\sigma}_{\vec{n}_i}) \leq N$$

convexity additivity bound for N spins

Therefore, if $F > N$ the state is entangled

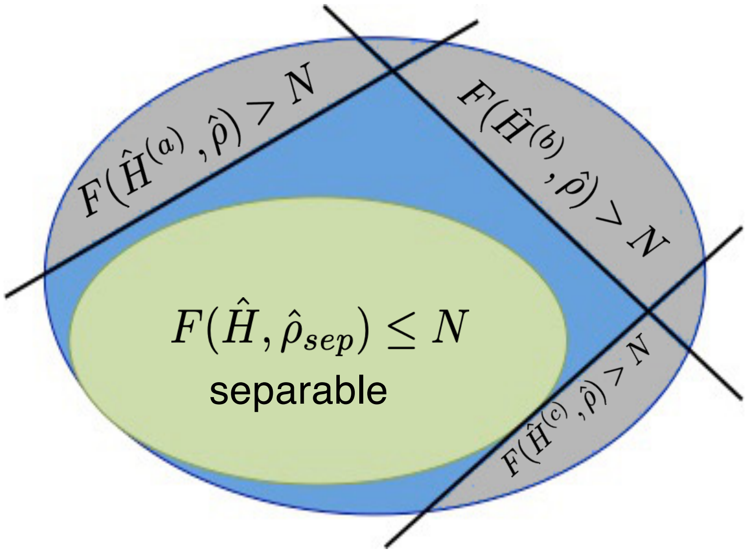
With a maximally entangled state (GHZ state) $F = N^2$

L. Pezzè and A. Smerzi, PRL 2009

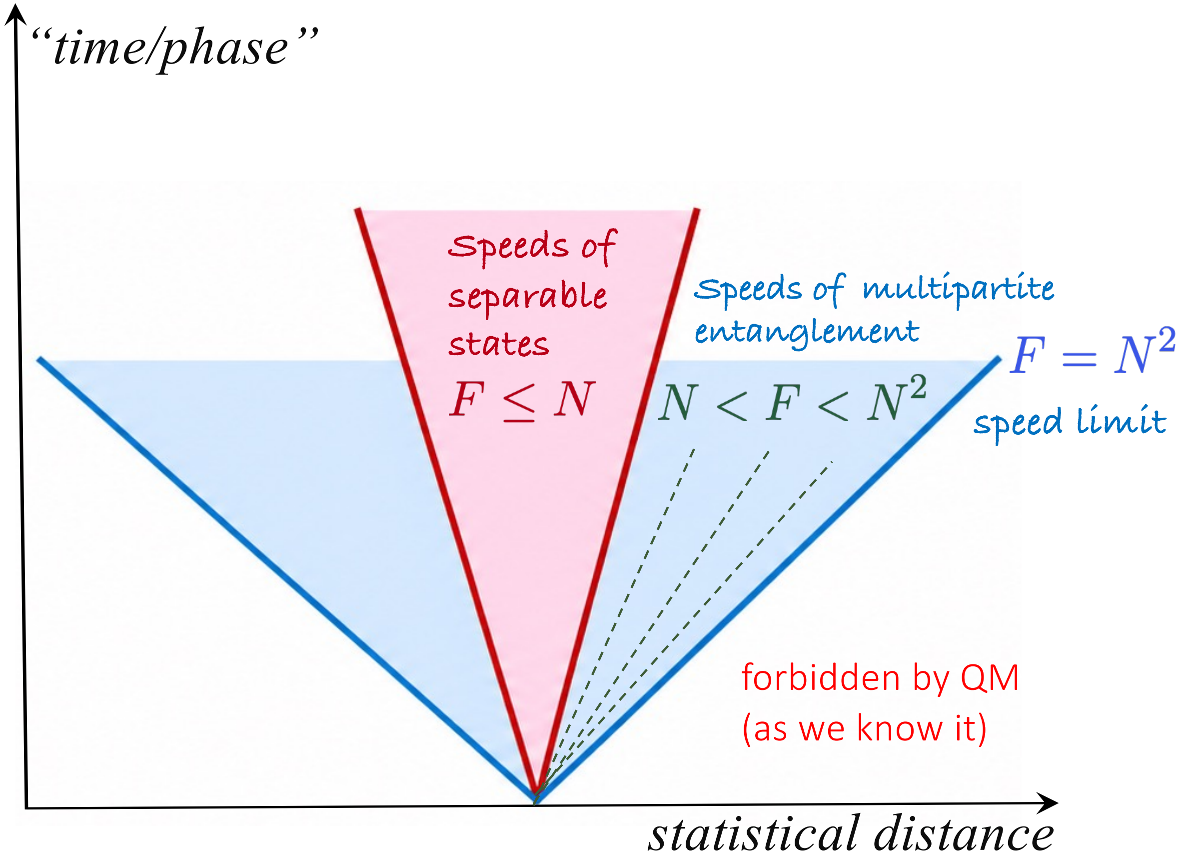
Intermediate values $N < F \leq N^2$ tell us about groups of entangled qubits

Fisher information links entanglement witnesses and speed limits

for local generators and unitary evolutions of qubit states



L. Pezzè and A. Smerzi, PRL, 2009



L.Pezzè, Y. Li, W. Li, & A.Smerzi, PNAS 2016
P. Hyllus et al., PRA **85**, 022321 (2012)
G. Tóth, PRA **85**, 022322 (2012)

MULTIPARTITE ENTANGLEMENT

Multipartite entanglement is not only richer than bipartite entanglement: its full characterization is computationally hard. Even deciding separability of general quantum states is NP-hard.

A classification scheme based on Young diagrams and integer indicators

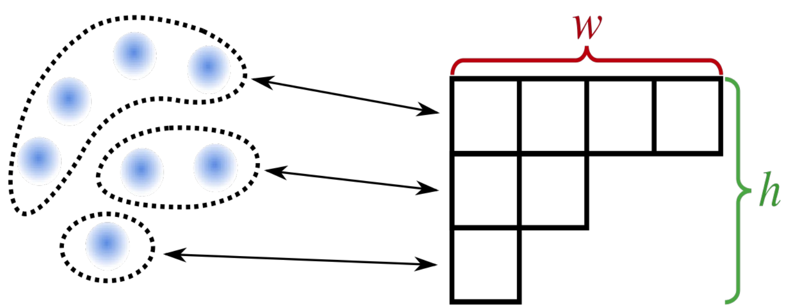
Example with a pure state of 7 qubits:

$\Lambda = \{4, 2, 1\}$ producibility

$$|\psi_7\rangle = |\psi_4\rangle|\psi_2\rangle|\psi_1\rangle$$

h=3 separability

w=4 entanglement depth



$$\Lambda = \{4, 2, 1\}$$

Entanglement depth w: largest group of entangled qubits
h-separability: the system can be split in h separable subsystems

$\Lambda-$producibility: the state can be written as a $\Lambda-$product state

Szalay, Quantum (2019)
Szalay, Toth, arXiv:2408.15350
Ren, Li, Smerzi, Gessner, PRL *2021*
Smerzi, Gessner, *unpublished*

Key point:

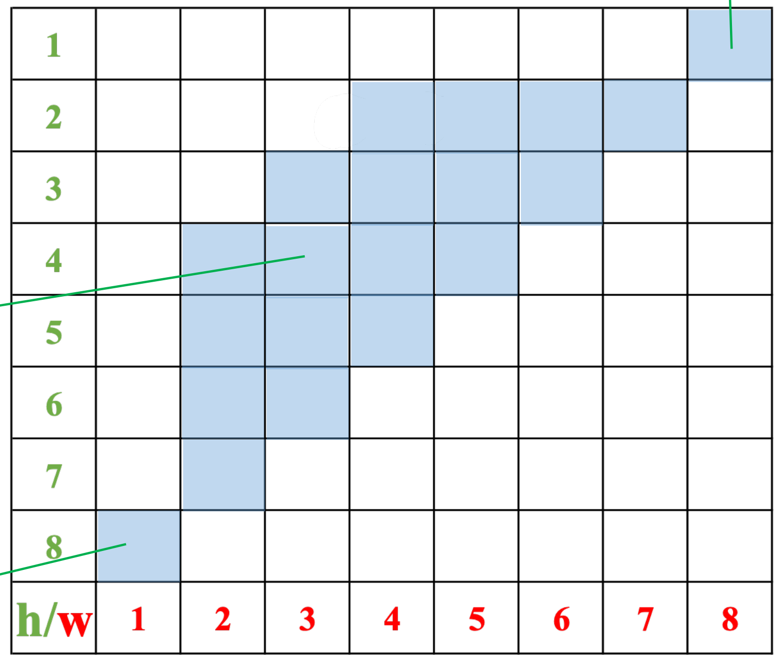
A mixed states can be written as are convex mixtures of different product-block structures.

$$\hat{\rho} = \sum_{\Lambda} p_{\Lambda} \; \hat{\rho}_{\Lambda}$$

maximum entanglement depth

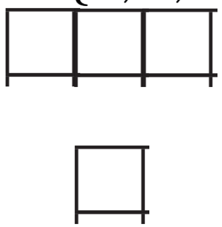
$\Lambda = \{8\}$

Example with N=8 particles



The blue squares contain allowed configurations given the N particles

$\Lambda = \{3, 2, 2, 1\}$



$\Lambda = \{1, 1, 1, 1, 1, 1, 1, 1\}$

fully separable

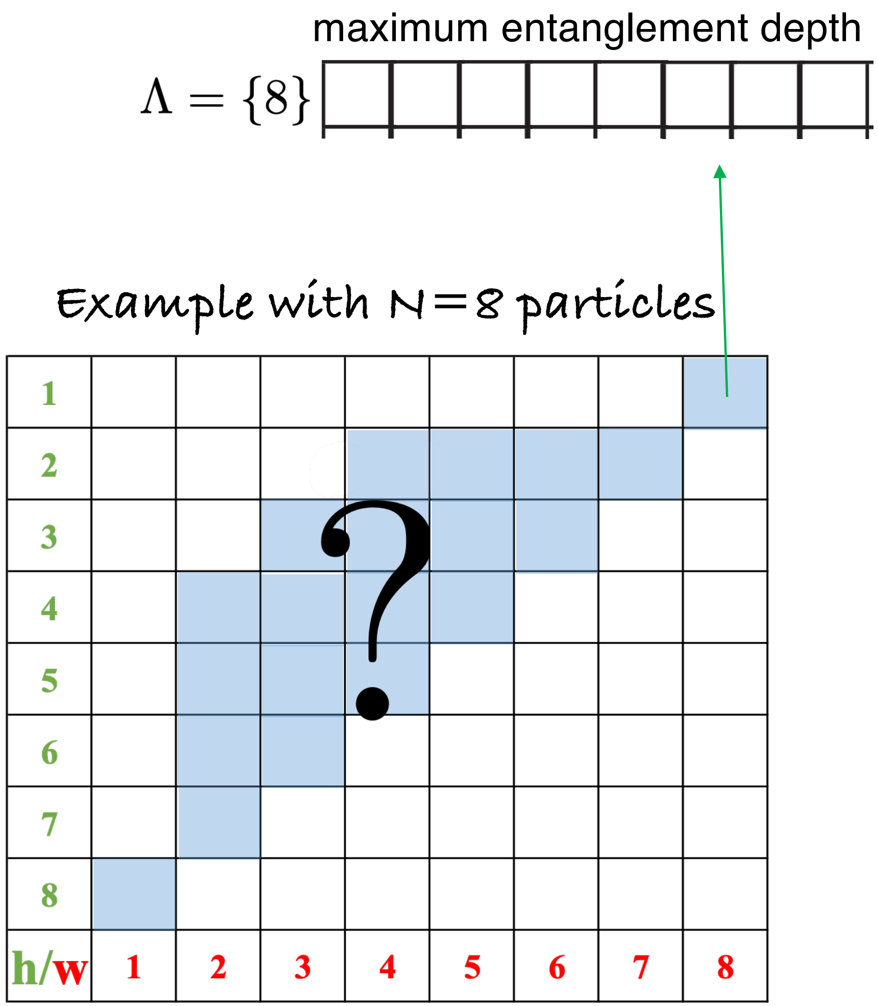
Key point:

A mixed states can be written as are convex mixtures of different product-block structures.

$$\hat{\rho} = \sum_{\Lambda} p_{\Lambda} \ \hat{\rho}_{\Lambda}$$

Can we experimentally determine the multipartite entanglement structure?

Certify the set of allowed parameters $\{p_{\Lambda}, \Lambda\}$ characterizing the state $\hat{\rho}$



The blue squares contain allowed configurations given the N particles

The protocol:

this is known!

$$F(\hat{\rho})_{exp} \leq \sum_{\Lambda} p_{\Lambda} \; F(\hat{\rho}_{\Lambda}) \leq \sum_{\Lambda} p_{\Lambda} \, F(\hat{\rho}_{\Lambda})_{max}$$

measure the
Fisher information

Example with N=8 particles and measured value
$F(\hat{\rho})_{exp} = 31$

1								64
2				32	34	40	50	
3			22	26, 24	30	38		
4		16	20, 18	22	28			
5		14	16	20				
6		12	14					
7		10						
8	8							
h/w	1	2	3	4	5	6	7	8

values of $F(\hat{\rho}_{\Lambda})_{max}$

The state cannot be written as a
mixture of red sectors

$$\hat{\rho} \neq \sum_{\Lambda \in red} p_{\Lambda} \; \hat{\rho}_{\Lambda}$$

the state must have support on at
least one blue Young-diagram sector.

The protocol:

Example with N=8 particles and measured value F = 31

1								64
2				32	34	40	50	
3			22	26, 24	30	38		
4		16	20, 18	22	28			
5		14	16	20				
6		12	14					
7		10						
8	8							
h/w	1	2	3	4	5	6	7	8

values of $F(\Lambda)_{max} = \sum_{i \in \Lambda} \lambda_i^2$

The state cannot be written as

$$\hat{\rho} \neq \sum_{\substack{\{\Lambda\} \\ F(\Lambda)_{max} < F}} p_{\Lambda} \; \hat{\rho}_{\Lambda}$$

The state cannot be written as a convex statistical distribution of the red squares. The state must include, both, blue and red squares with statistical weights constrained by the measured value of the Fisher information

Sufficient condition to exclude a density matrix equal to the convex sum of a set of $\Lambda-$ groups

How can F_Q be accesses experimentally?

No full tomography is needed: measured observables can lower-bound

$$F_Q \geq \sum_{\mu} \frac{1}{P(\mu|\phi)} \left(\frac{\partial P(\mu|\phi)}{\partial \phi}\right)^2 \geq \frac{|\langle[\hat{H},\hat{\mu}]\rangle|^2}{\Delta^2(\hat{\mu})}$$

quantum Fisher

"classical" Fisher

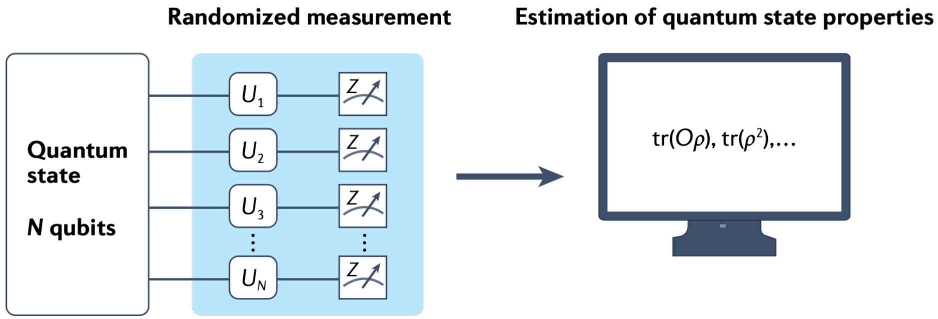
spin squeezing

Wineland, et al. PRA (1994)
Sorensen, Duan, Cirac, Zoller, NATURE (2001)
Pezzè, Smerzi, Varenna school, 2017

$$F_{\mathrm{Q}}(T) = \frac{4}{\pi} \int_0^\infty \mathrm{d}\omega \tanh\left(\frac{\omega}{2T}\right) \chi''(\omega, T)$$

Hauke, Heyl, Tagliacozzo, Zoller, Nat. Phys. (2016)

$\chi''(\omega, T)$ is the imaginary dissipative part of the dynamical susceptibility



Bound using the randomized measurements family.
For a review see: Elben et al., Nature Reviews Physics (2023)

Quantum spin dynamics and entanglement generation with hundreds of trapped ions

Justin G. Bohnet[1,*], Brian C. Sawyer[1,2], Joseph W. Britton[1,3], Michael L. Wall[4], Ana Maria Rey[5], Michael Foss-Feig[3,6], John J. Bollinger[1,*]

Twin Matter Waves for Interferometry Beyond the Classical Limit

B. Lücke,[1*] M. Scherer,[1*] J. Kruse,[1] L. Pezzé,[2] F. Deuretzbacher,[3] P. Hyllus,[4] O. Topic,[1] J. Peise,[1] W. Ertmer,[1] J. Arlt,[5] L. Santos,[3] A. Smerzi,[2] C. Klempt[1]†

Science AAAS

Fisher information and entanglement of non-Gaussian spin states

Helmut Strobel,[1*] Wolfgang Muessel,[1] Daniel Linnemann,[1] Tilman Zibold,[1] David B. Hume,[1] Luca Pezzè,[2] Augusto Smerzi,[2] Markus K. Oberthaler[1]

nature physics
LETTERS
PUBLISHED ONLINE: 6 JUNE 2016 | DOI: 10.1038/NPHYS3783

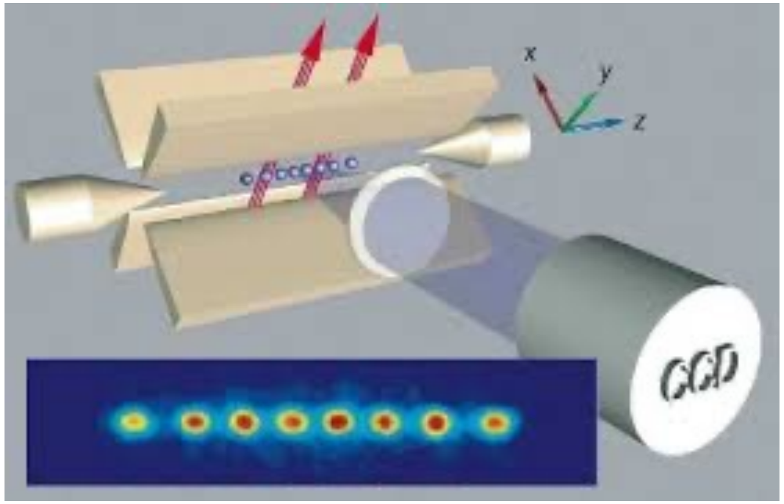
Many-body localization in a quantum simulator with programmable random disorder

J. Smith[1*], A. Lee[1], P. Richerme[2], B. Neyenhuis[1], P. W. Hess[1], P. Hauke[3,4], M. Heyl[3,4,5], D. A. Huse[6] and C. Monroe[1]

Hong-Ou-Mandel interference of more than 10 indistinguishable atoms

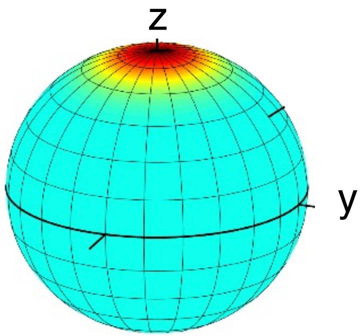
Martin Quensen[1,2*], Mareike Hetzel[1,2], Luis Santos[1], Augusto Smerzi[3,4,5], Géza Tóth[6,7,8,9,10], Luca Pezzè[3,4,5], Carsten Klempt[1,2]

EXAMPLE: MULTIPARTITE ENTANGLEMENT WITH TRAPPED IONS



The ions are initially prepared in the product state:

$$|1\rangle^{\otimes N} = |111\cdots 1\rangle$$

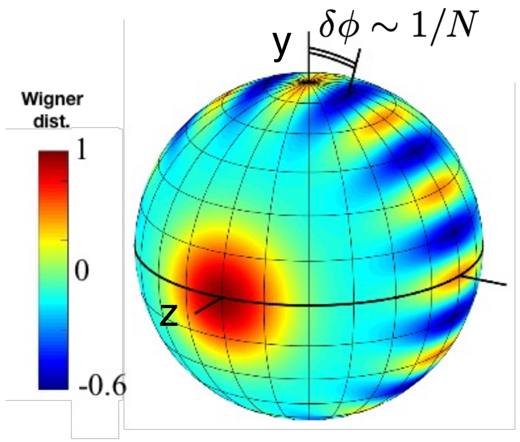


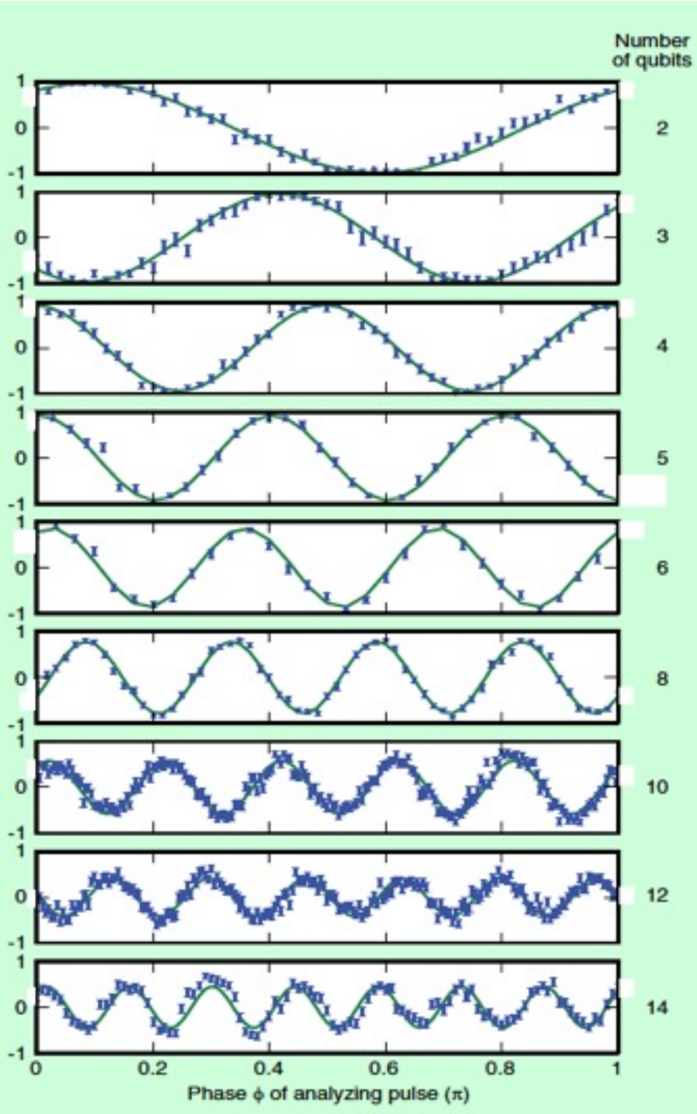
2) Apply a Mølmer–Sørensen gate (global entangling operation)

$$U_{MS} = \exp\left(-i\frac{\pi}{4}\left(\sum_{j=1}^{N}\sigma_x^{(j)}\right)^2\right)$$

Then the GHZ state is generated by:

$$U_{MS}\cdot U_R\,|1\rangle^{\otimes N} = \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}\right) = |\mathrm{GHZ}\rangle$$





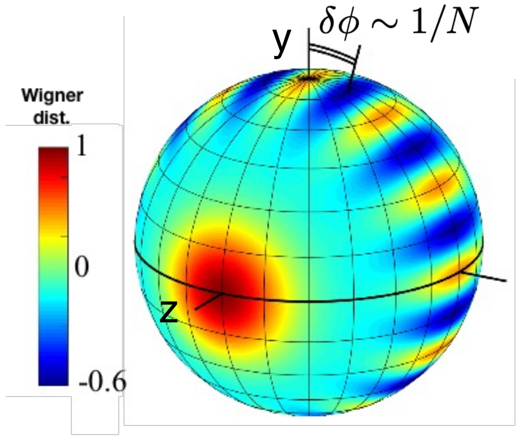
Monz et al PRL 2011 (Innsbruck)

A phase ϕ is encoded via a rotation around the z-axis:

$$\hat{U}(\phi) = e^{-iJ_z\phi}, \qquad \text{with } J_z = \frac{1}{2}\sum_{k=1}^{N}\sigma_z^{(k)}$$

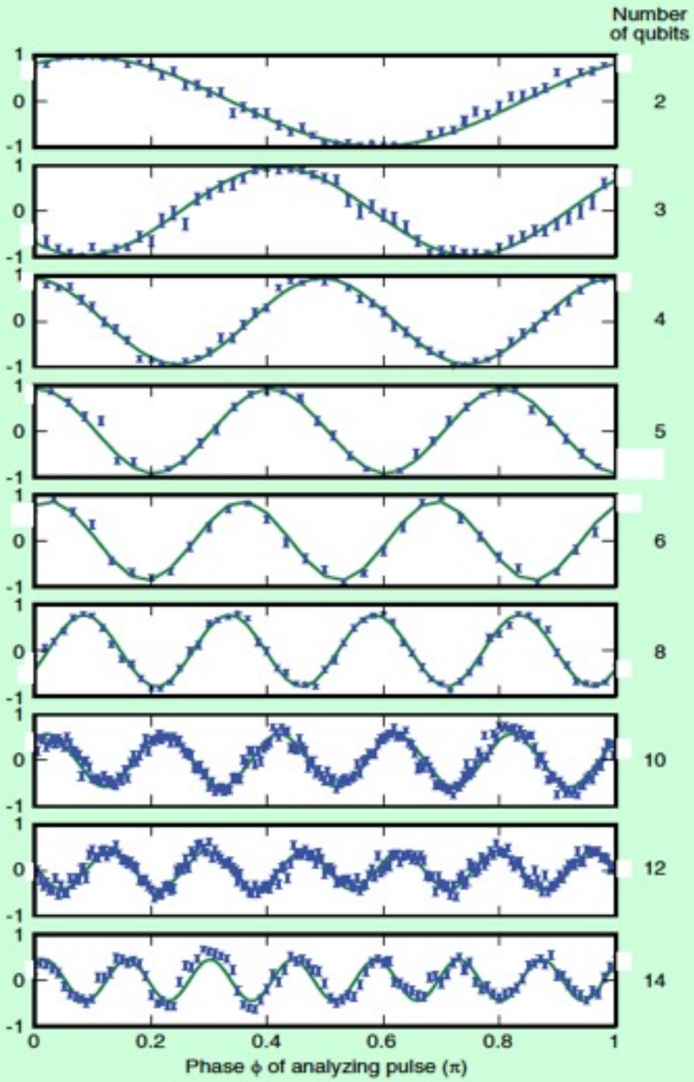
This evolves the GHZ state to:

$$|\psi(\phi)\rangle = \hat{U}(\phi)\,|GHZ\rangle = \frac{1}{\sqrt{2}}\left(e^{-iN\phi/2}\,|0\rangle^{\otimes N} + e^{iN\phi/2}\,|1\rangle^{\otimes N}\right)$$



and get the oscillating functions:

$$P(\phi) = Tr[\hat{\rho}_{exp}\;\hat{\Pi}] = \frac{1 + V\cos(N\phi)}{2}$$



Monz et al PRL 2011 (Innsbruck)

Measurement protocol:

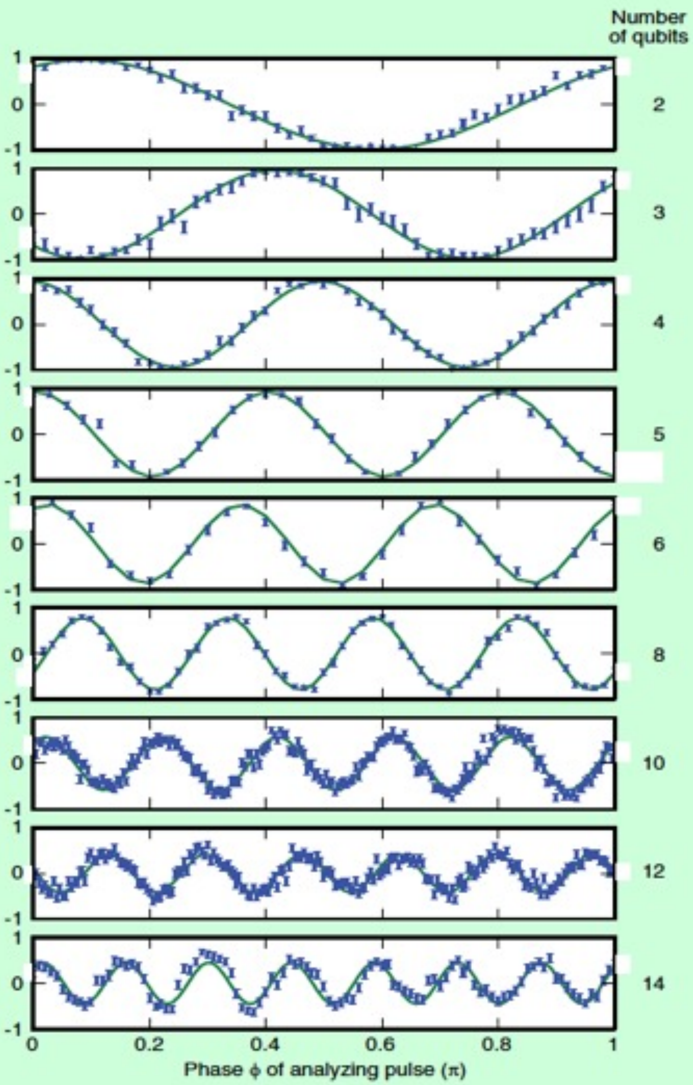
first counter-rotate the state with: $\exp\left(+i\frac{\pi}{4}\sum_{j=1}^{N}\sigma_y^{(j)}\right)$

than perform a parity measurement: $\hat{\Pi} = \prod_{j=1}^{N}\sigma_z^{(j)} = (-1)^{N_{up}}$

and get the oscillating functions:

$$P(\phi) = Tr[\hat{\rho}_{exp}\ \hat{\Pi}] = \frac{1 + V\cos(N\phi)}{2}$$

Notice that the frequency of the oscillations increases with the number of ions N but the visibility V decreases with N due to decoherence effects



Monz et al PRL 2011 (Innsbruck)

Question:
Can we certify that the state generated experimentally is really entangled?
Calculate the Fisher information:

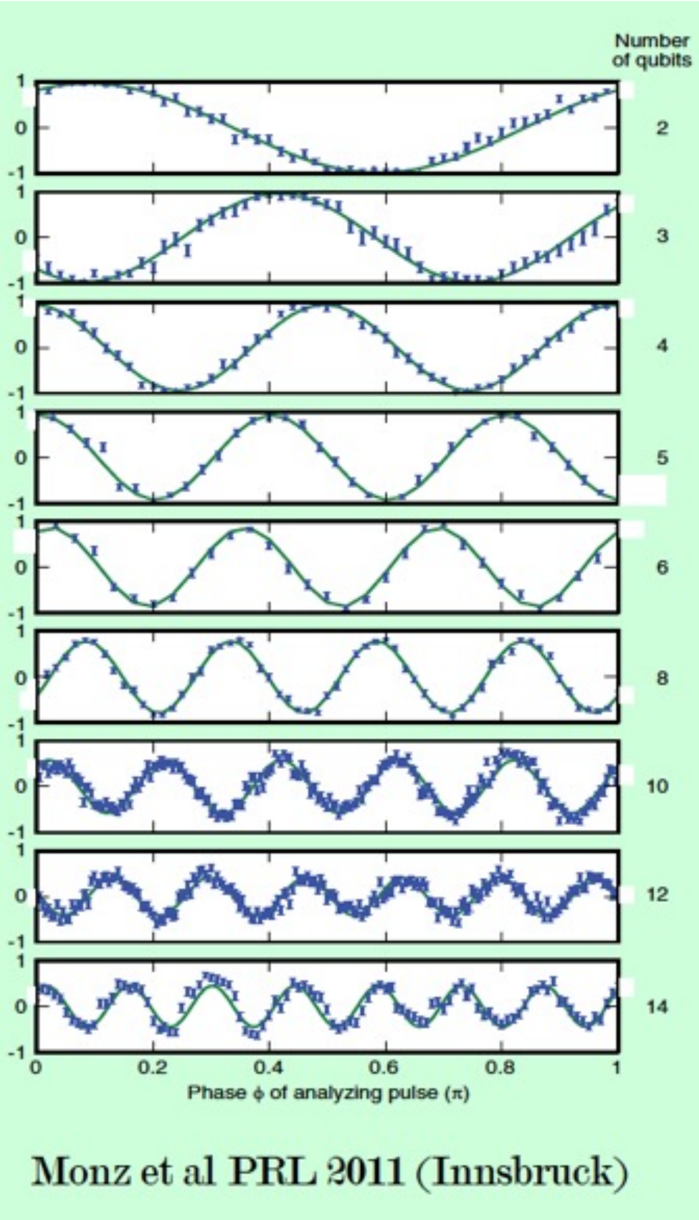
Consider a two-outcome measurement with probabilities:

$$p_1(\phi) = P(\phi), \qquad p_2(\phi) = 1 - P(\phi)$$

The classical Fisher information is given by:

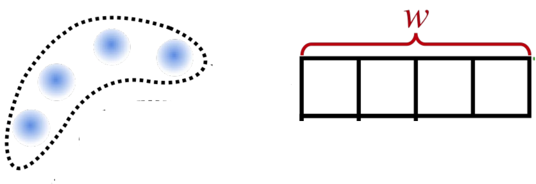
$$F(\phi) = \sum_{i=1}^{2} \frac{1}{p_i(\phi)} \left(\frac{\partial p_i(\phi)}{\partial \phi} \right)^2$$

$$F(\phi) = \frac{V^2 N^2 \sin^2(N\phi)}{1 - V^2 \cos^2(N\phi)}$$

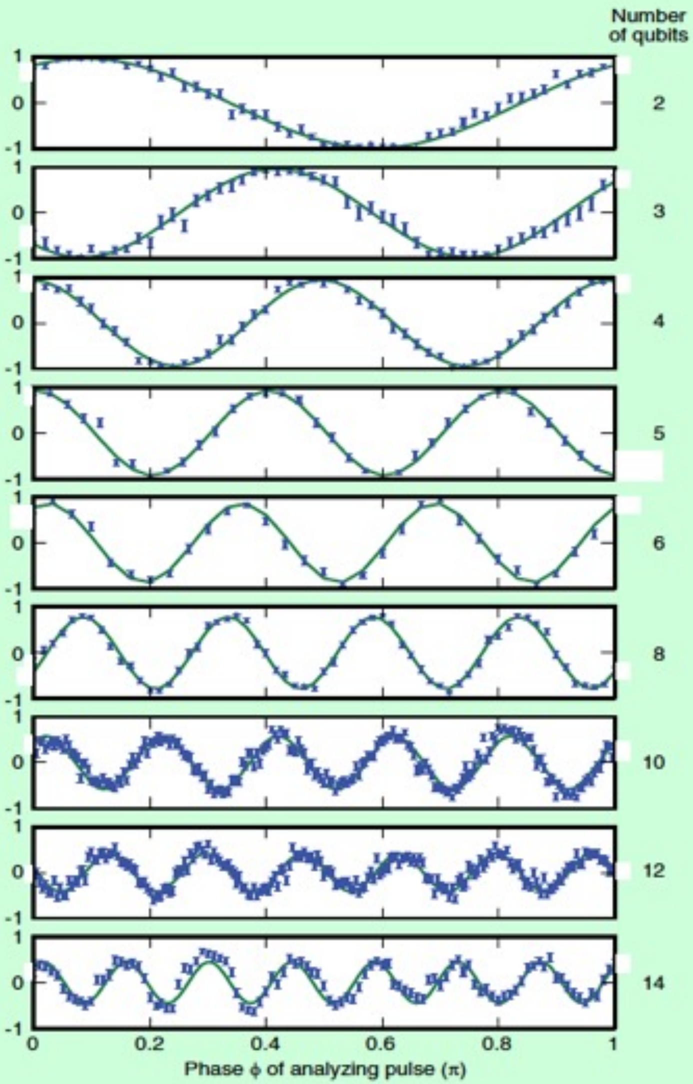


Question:
Is the state generated experimentally really entangled?
Calculate the Fisher information:

from exp data with N=4 we obtain F=16. This value certifies the creation of a GHZ state of 4 qubits



$w = 4, h = 1$



Monz et al PRL 2011 (Innsbruck)

with N=14 we obtain F=40.4

1								64
2				32	34	40	50	
3			22	26, 24	30	38		
4		16	20, 18	22	28			
5		14	16	20				
6		12	14					
7		10						
8	8							
h/w	1	2	3	4	5	6	7	8

blue blocks are compatible with $F = 40.4$

The state cannot be written as a convex combination of red blocks

Note: even with visibility V < ½, multipartite entanglement can be certified

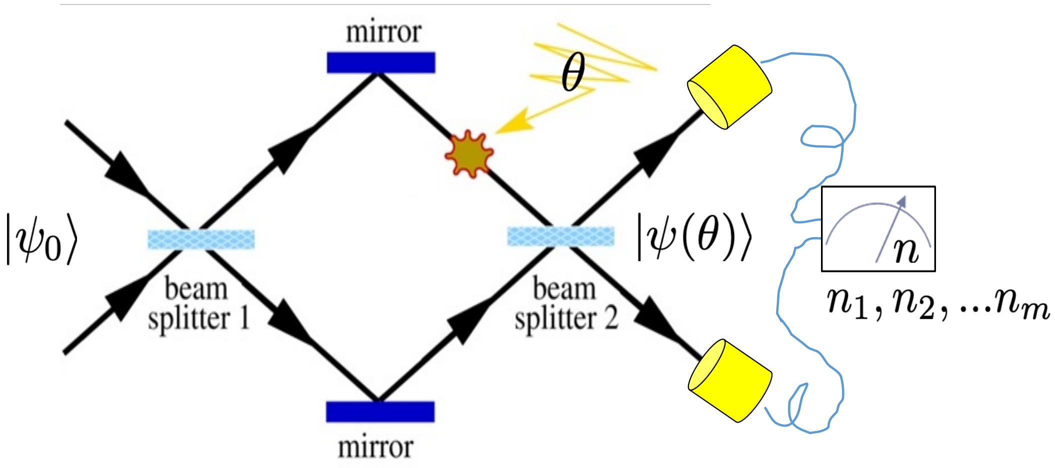
Quantum phase estimation: what is it?

state of the particles entering in the interferometer

$$|\psi(\theta)\rangle = e^{-i\hat{H}\theta}|\psi_0\rangle$$

phase

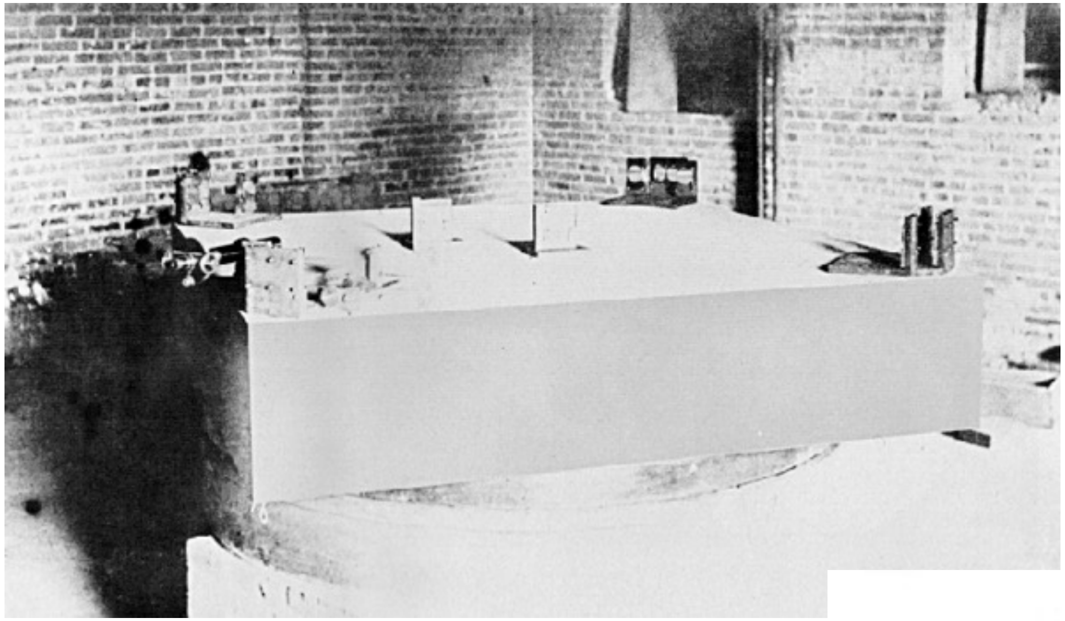
Hamiltonian of the interferometer



With a set of measurement results $n_1, n_2, \ldots n_m$ obtained at an unknown θ with probability $P(n|\theta) = |\langle n|\psi(\theta)\rangle|^2$

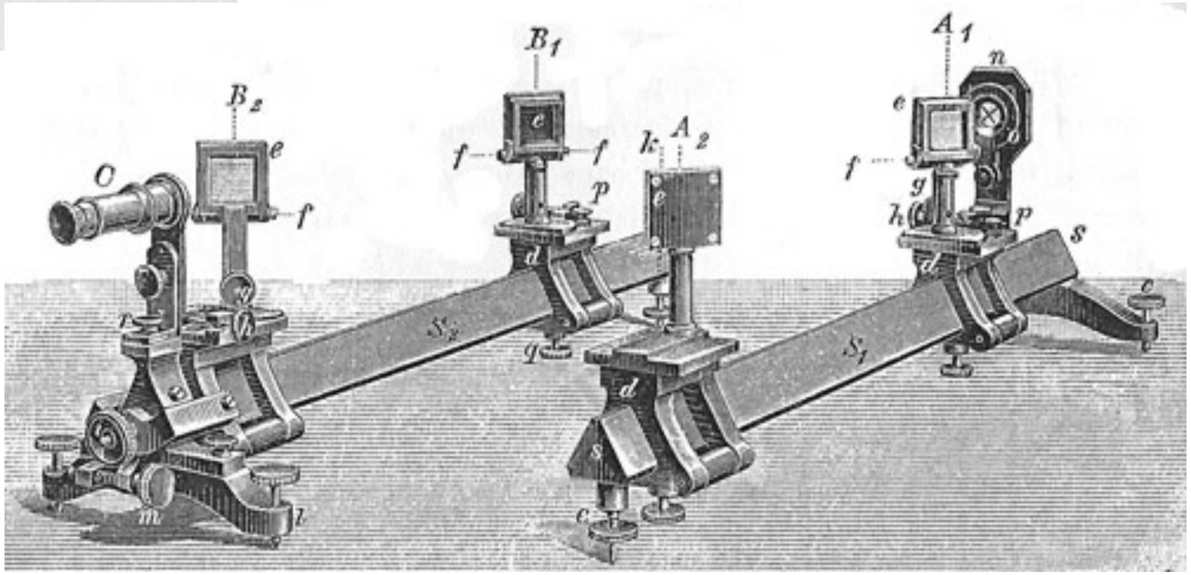
Goal: estimate θ

The value of the phase depends on the interaction of an external classical field with the quantum particles entering in the interferometer. The estimation of the value of the phase provides information on the strength of the field. This allows to detect and measure, for instance, magnetic fields, gravitational waves, etc.



Michelson and Morley (1887)

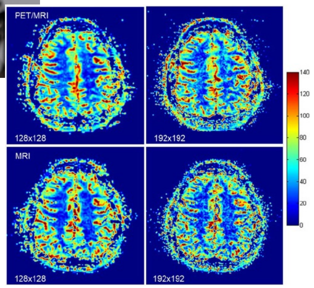
The pre-history of quantum interferometry



L. Zehnder, Zeits (1891)
L. Mach, (1892)
E. Mach, L. Mach (1889)

Modern platforms and applications

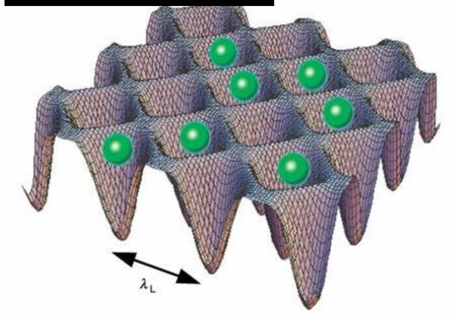
non-invasive diagnostics
(EEG, MEG…)



imaging

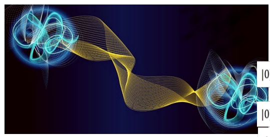
accelerometers and
gyroscopes for navigation

atomic clocks
and clock synchronization

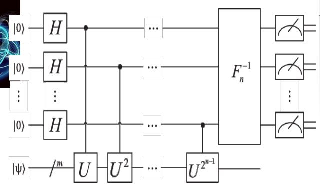


network of GW interferometers

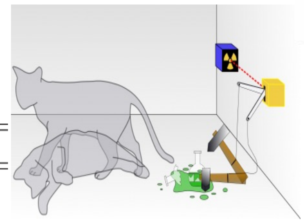
interdisciplinary theory



quantum information



computing

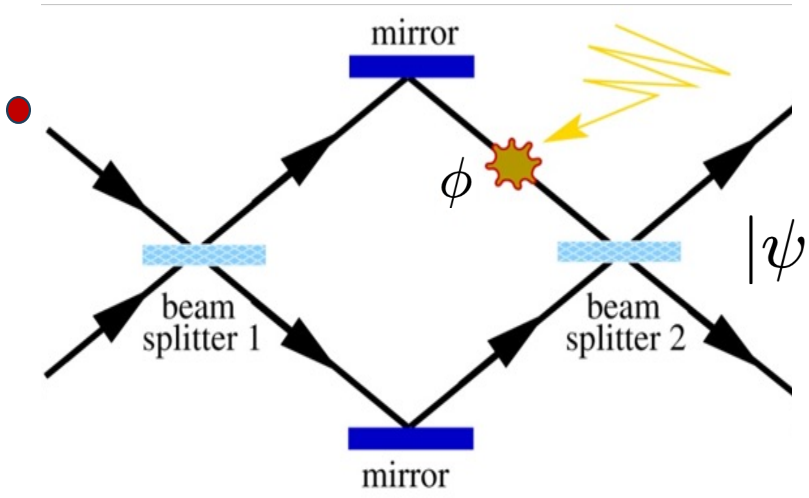


foundations

MACH-ZEHNDER INTERFEROMETRY WITH QUBITS

A SINGLE QUBIT CAN BE PHYSICALLY REALIZED WITH ONE PARTICLE AND A MZ INTERFEROMETER

particle entering in
upper port: spin up $|\uparrow\rangle$



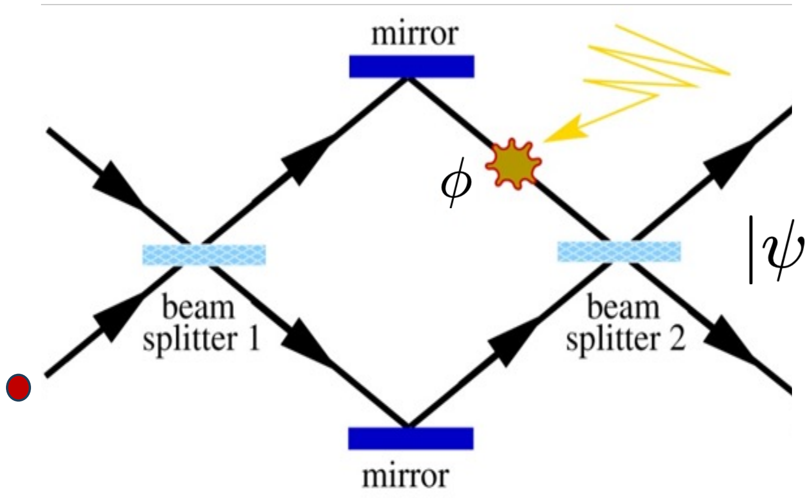
$|\psi(\phi)\rangle$

The MZ interferometer can drive the qubit around the Bloch sphere

A SINGLE QUBIT CAN BE PHYSICALLY REALIZED WITH ONE PARTICLE AND A MZ INTERFEROMETER

Example: particle entering in lower port: spin down

$|\downarrow\rangle$

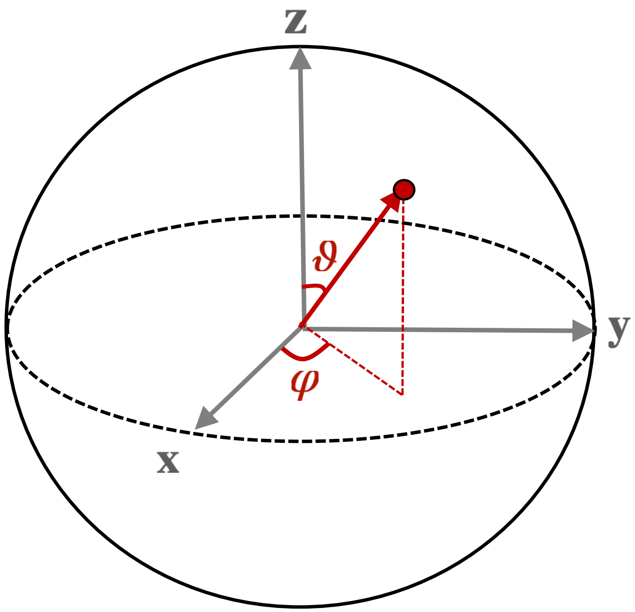


$|\psi(\phi)\rangle$

The MZ interferometer can drive the qubit around the Bloch sphere

Bloch sphere representation of a qubit

Bloch sphere:
every pure state is a point on the surface



general pure state of a qubit

$$|\psi\rangle = \cos\frac{\vartheta}{2}|\uparrow\rangle + e^{i\varphi}\sin\frac{\vartheta}{2}|\downarrow\rangle$$

Pauli matrices:

$$\hat{\sigma}_x = |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|$$

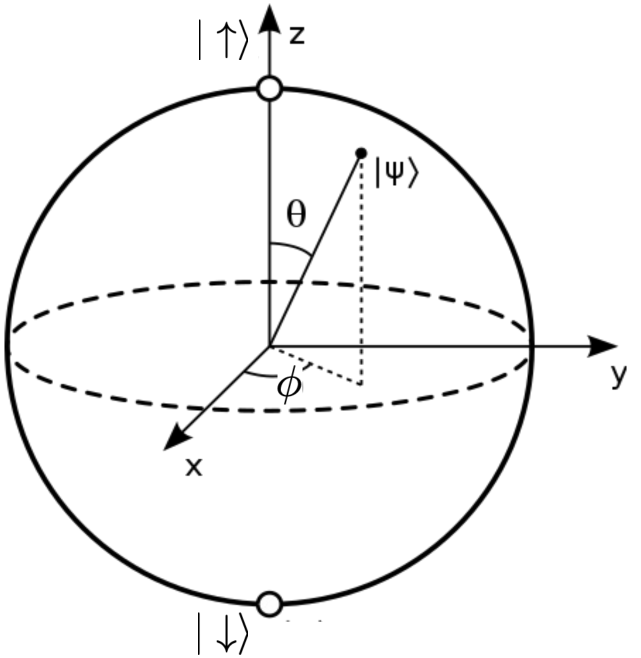
$$\hat{\sigma}_y = i(|\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow|)$$

$$\hat{\sigma}_z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$[\hat{\sigma}_x, \hat{\sigma}_y] = i\hat{\sigma}_z \qquad \hat{\sigma}^2_{x,y,z} = \hat{1}$$

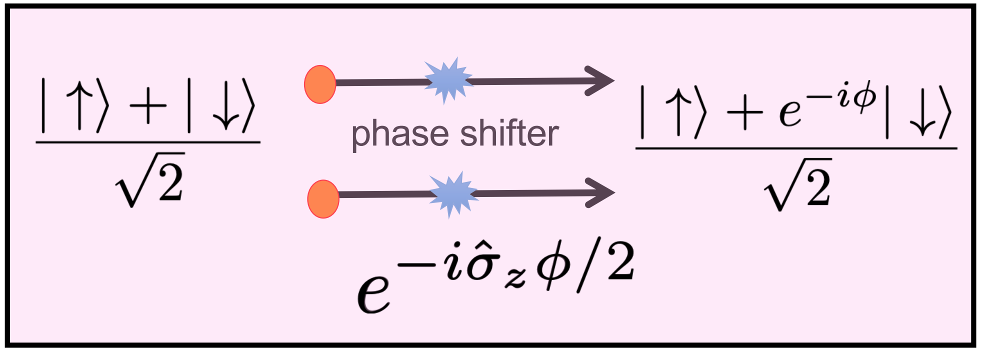
see for instance Nielsen & Chuang *Quantum Computation and Quantum information* (2001)

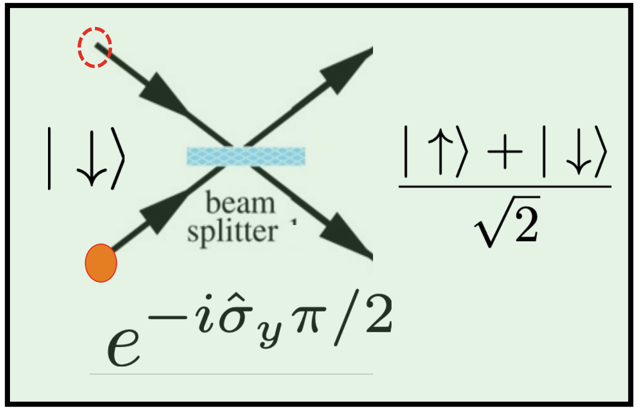
How to drive a qubit around the Bloch sphere



$$e^{-i\gamma\hat{\sigma}_{\vec{n}}/2} = \cos\frac{\gamma}{2} - i\hat{\sigma}_{\vec{n}}\sin\frac{\gamma}{2}$$

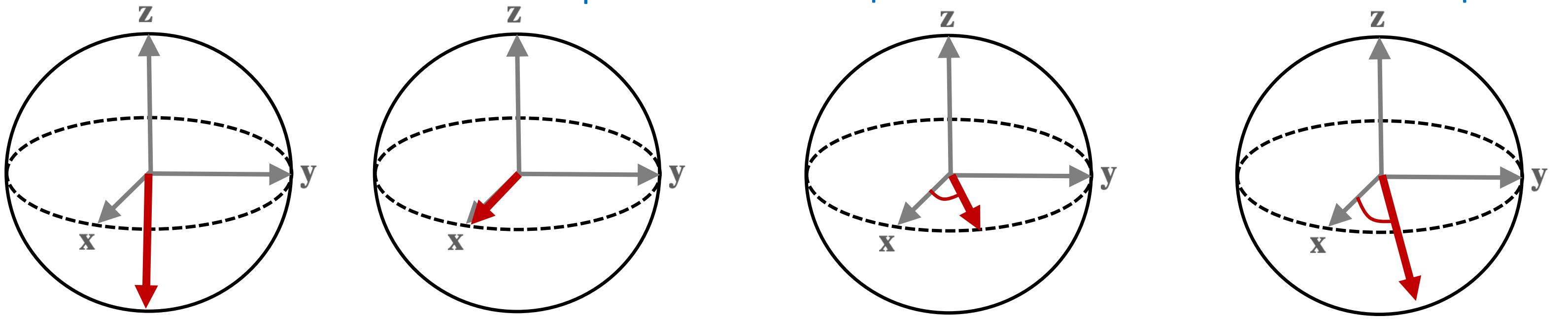
every unitary operator is a rotation:
rotate the state around the $\vec{n}$ axis by an angle γ





MACH-ZEHNDER INTERFEROMETER AND BLOCH SPHERE

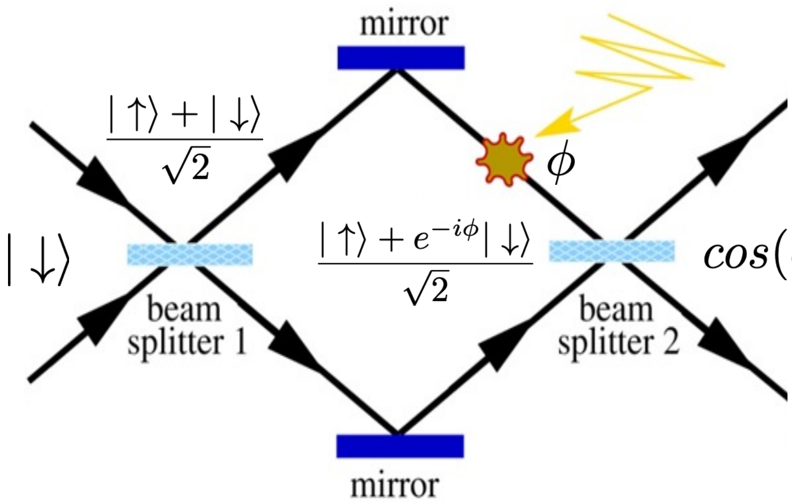
initial state | first beam splitter | phase shift | second beam splitter



$|\downarrow\rangle$ $\quad e^{-i\hat{\sigma}_y\pi/2}$ $\quad e^{i\hat{\sigma}_z\phi/2}$ $\quad e^{i\hat{\sigma}_y\pi/2}$

MZ evolution of a qubit

$$= e^{-i\hat{\sigma}_x\phi/2}|\downarrow\rangle$$



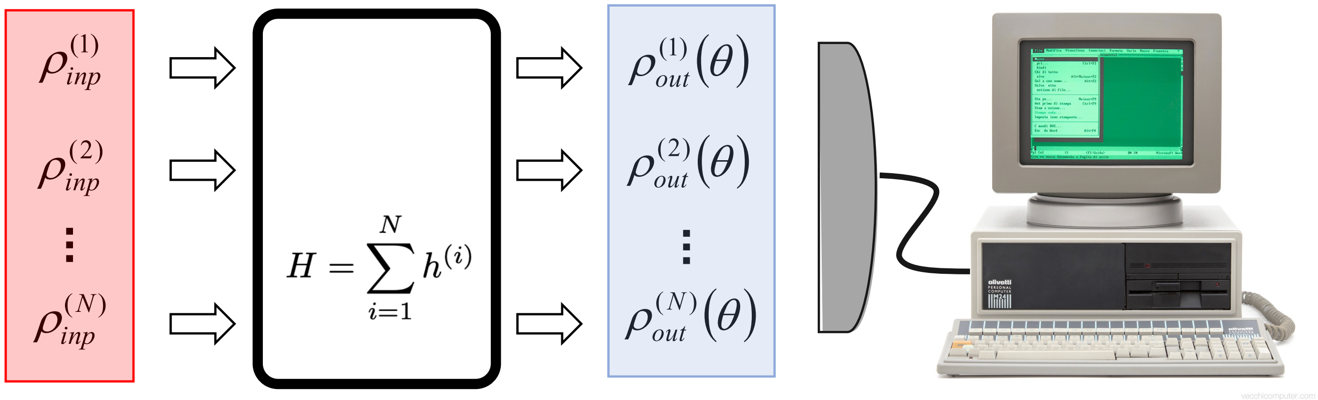
$$cos(\phi/2)|\uparrow\rangle + sin(\phi/2)|\downarrow\rangle$$

Measurement probabilities depend on the phase shift

$$P(\uparrow|\phi) = |\langle\uparrow|e^{-i\hat{\sigma}_x\phi/2}|\uparrow\rangle|^2 = cos^2(\phi/2)$$

$$P(\downarrow|\phi) = |\langle\downarrow|e^{-i\hat{\sigma}_x\phi/2}|\uparrow\rangle|^2 = sin^2(\phi/2)$$

INTERFEROMETRY WITH N QUBITS



Input state → Evolution → Measurement

$$\rho$$

$$\rho(\theta) = U(\theta)\rho U^{\dagger}(\theta)$$

$$p(\mu|\theta) = \mathrm{Tr}\{\rho(\theta)\hat{\mu}\}$$
$$\{\mu\}_m = \{\mu_1, \ldots, \mu_m\}$$

Comments:

i) The input state is represented here a generic density matrix.
ii) The state can be separable or entangled in the qubits degrees of freedom.
iii) The evolution of each qubit is governed by a Pauli matrix

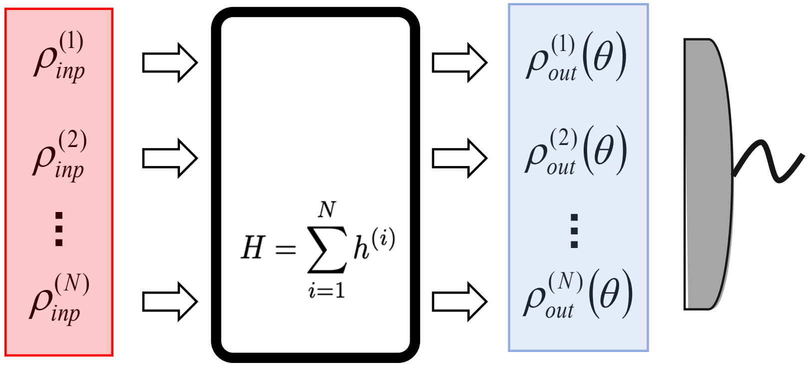
INTERFEROMETRY WITH N QUBITS -> TWO MODES MACH-ZEHNDER

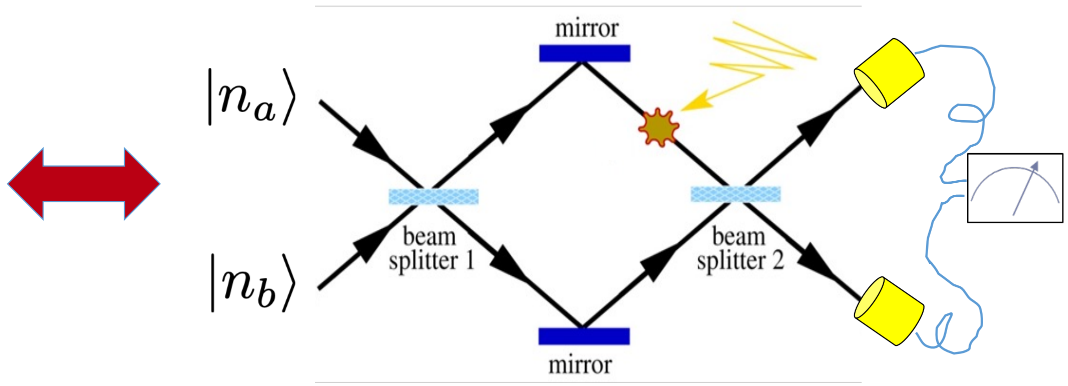
Consider:

i) symmetric input (with respect to the exchange of qubits labels)

ii) symmetry preserving transformation (possible with collective rotations)

--> "easily" realizable experimentally with indistinguishable bosons.





Dimension of the Hilbert space with N qubits and N interferometers: 2^N

Dimension of the Hilbert space with $N = n_a + n_b$ particles and 1 interferometer: $N + 1$

MACH-ZEHNDER INTERFEROMETER

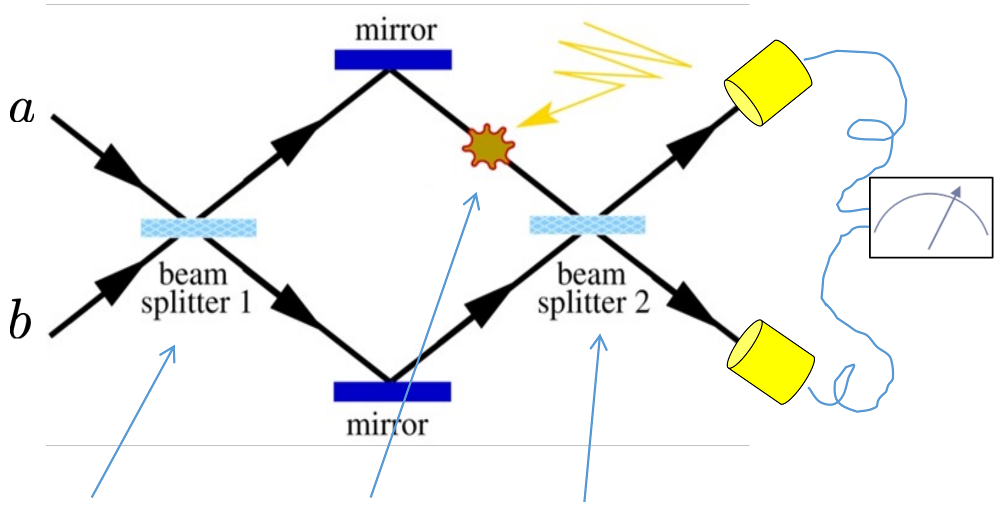
Schwinger (pseudo-)angular momentum representation of the symmetric subspace

$$\hat{J}_x = \frac{1}{2}\sum_{i=1}^{N}\hat{\sigma}_x^{(i)} \equiv \frac{\hat{a}\hat{b}^\dagger + \hat{a}^\dagger\hat{b}}{2}$$

$$\hat{J}_y = \frac{1}{2}\sum_{i=1}^{N}\hat{\sigma}_y^{(i)} \equiv \frac{\hat{a}^\dagger\hat{b} - \hat{a}\hat{b}^\dagger}{2i}$$

$$\hat{J}_z = \frac{1}{2}\sum_{i=1}^{N}\hat{\sigma}_z^{(i)} \equiv \frac{\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}}{2}$$

$$[\hat{J}_x, \hat{J}_y] = i\hat{J}_z$$

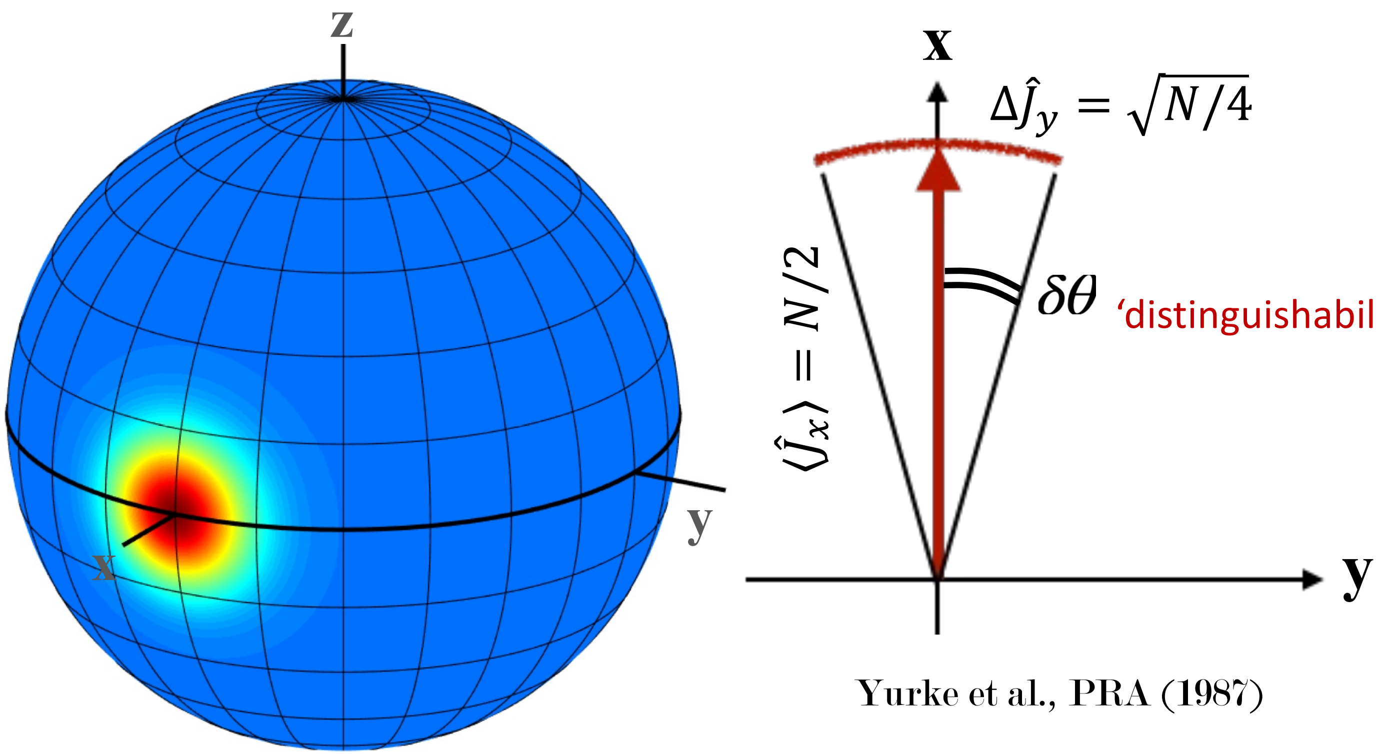


$$e^{-i\hat{J}_y\frac{\pi}{2}}\; e^{-i\hat{J}_z\phi}\; e^{i\hat{J}_y\frac{\pi}{2}} = e^{-i\hat{J}_x\phi}$$

$\hat{a}, \hat{a}^\dagger$ creation, destruction operators

The Mach-Zehnder rotates the state around in a generalized Bloch-sphere.

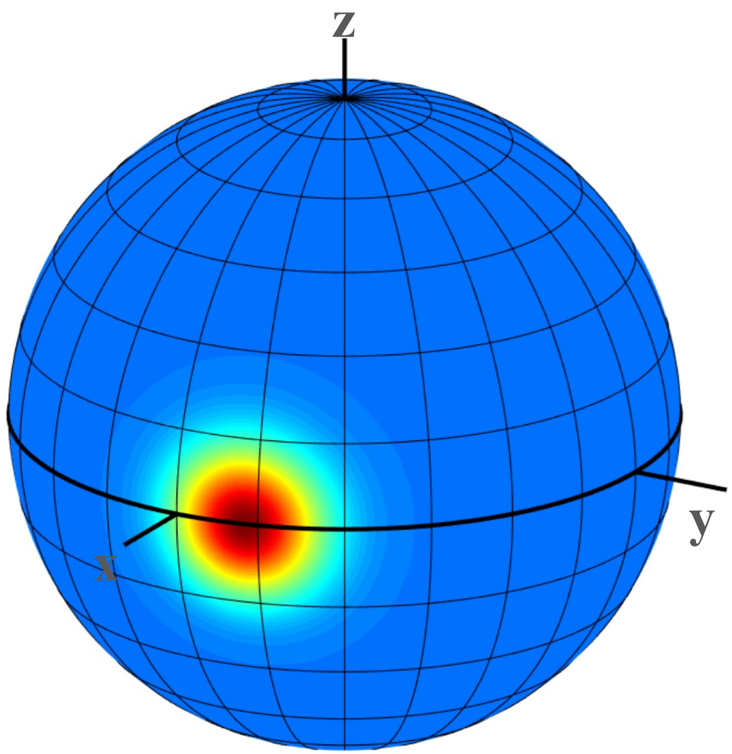
Sensitivity is proportional to the minimum displacement
such the the final state becomes orthogonal to the initial one.

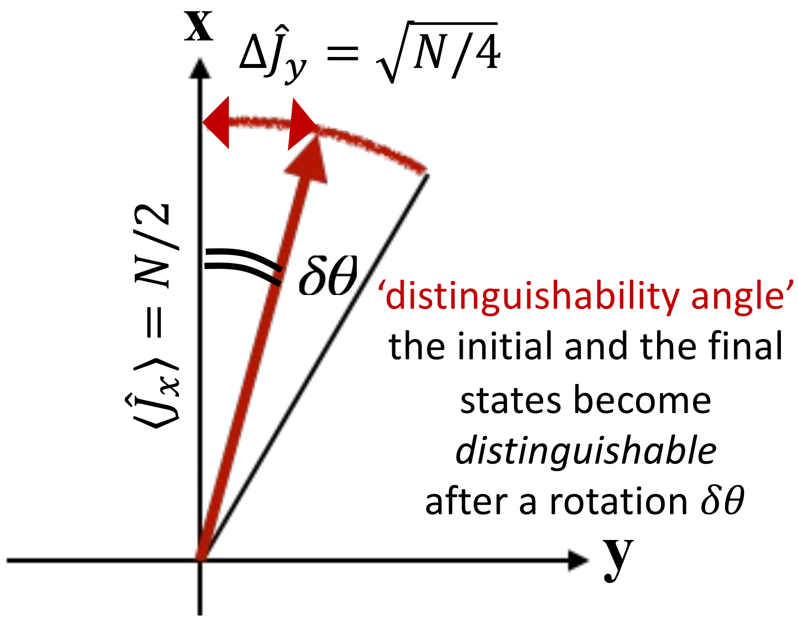


COHERENT SPIN STATE

The Mach-Zehnder rotates the state around in a generalized Bloch-sphere.

Sensitivity is proportional to the minimum displacement
such the the final state becomes orthogonal to the initial one.





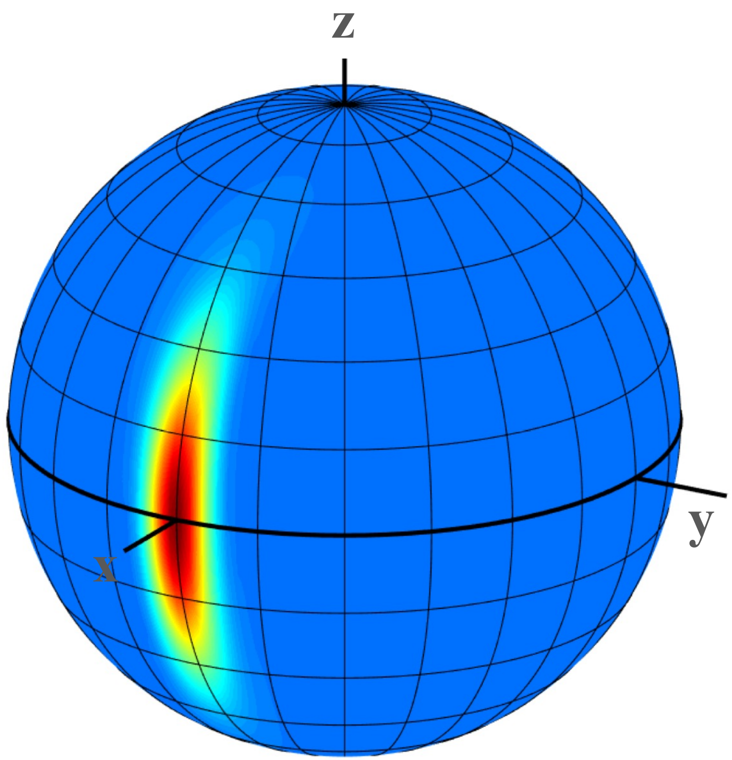
Yurke et al., PRA (1987)

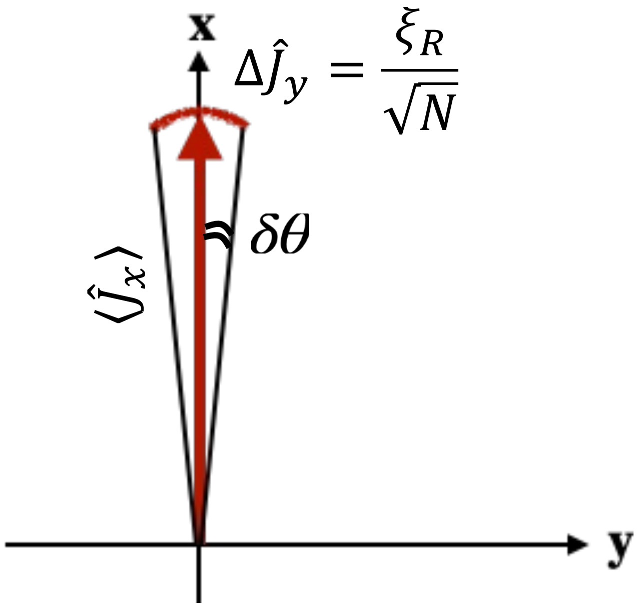
COHERENT SPIN STATE

$$\delta\theta^2 = \frac{\Delta^2 \hat{J}_y}{\langle \hat{J}_x \rangle^2} = \frac{1}{N}$$ «shot-noise»

The Mach-Zehnder rotates the state around in a generalized Bloch-sphere.

Sensitivity is proportional to the minimum displacement
such the the final state becomes orthogonal to the initial one.

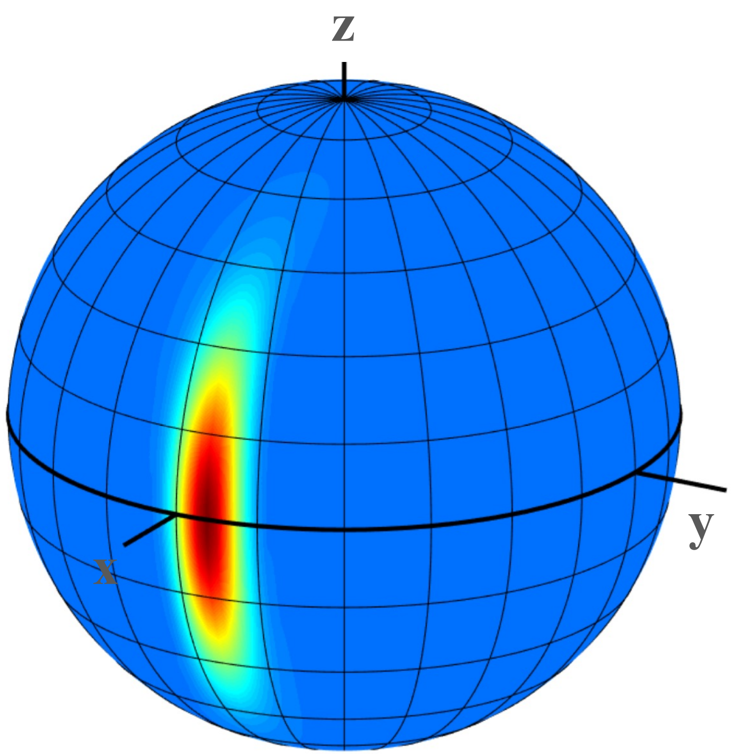




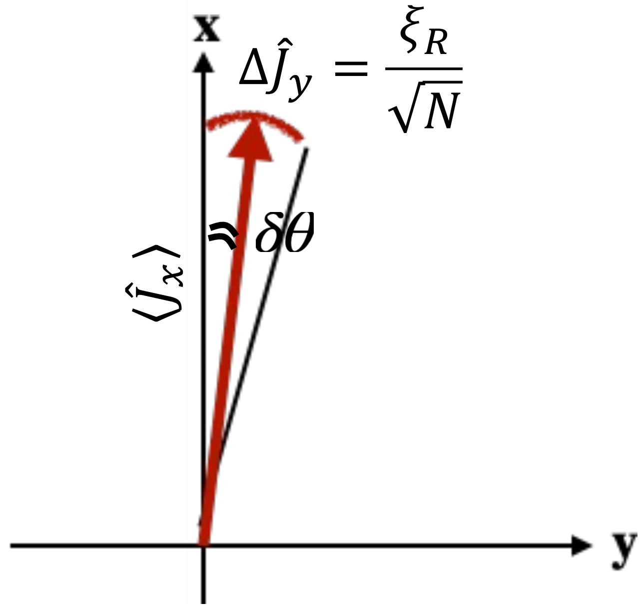
SQUEEZED SPIN STATE

The Mach-Zehnder rotates the state around in a generalized Bloch-sphere.

Sensitivity is proportional to the minimum displacement such the the final state becomes orthogonal to the initial one.



SQUEEZED SPIN STATE

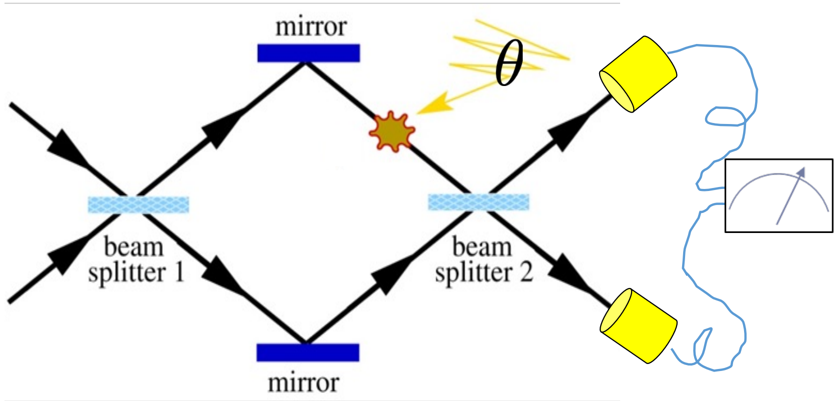


$$\delta\theta = \frac{\Delta\hat{J}_y}{\langle\hat{J}_x\rangle} = \frac{\xi_R}{\sqrt{N}}$$

if $\xi_R < 1$ then: $(\delta\theta)^2 < \frac{1}{N}$ «sub shot-noise»

QUANTUM PHASE ESTIMATION THEORY

INTERFEROMETRY TURNS PHASE INTO DATA



collect the results of m measurements

$$\mu_1, \mu_2, \ldots \mu_m$$

$$\theta \longrightarrow p(\mu|\theta) \longrightarrow \mu_1, \mu_2, \ldots \mu_m \longrightarrow \hat{\theta}_m$$

true phase — outcome distribution — repeated data — phase estimate

A good estimators

$$\mathbb{E}_\theta[\hat{\theta}_m] = \theta$$

$$\hat{\theta}_m \to \theta \quad \text{as} \quad m \to \infty$$

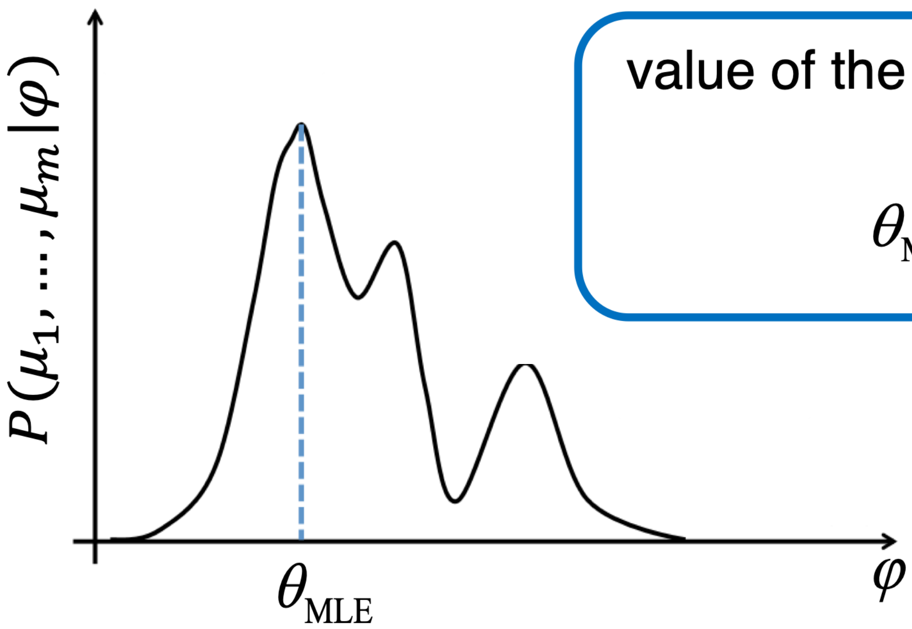
Metrology: how small can the phase uncertainty become?

$$\Delta^2 \hat{\theta}_m = \mathbb{E}_\theta[(\hat{\theta}_m - \theta)^2]$$

The Maximum likelihood estimator

It uses the probability of the outcomes as function of the phase $P(\mu_1 \ldots \mu_m|\phi) = \prod_{i=1}^{m} P(\mu_i|\phi)$:

Given the observed outcomes, choose the phase that makes them most probable.



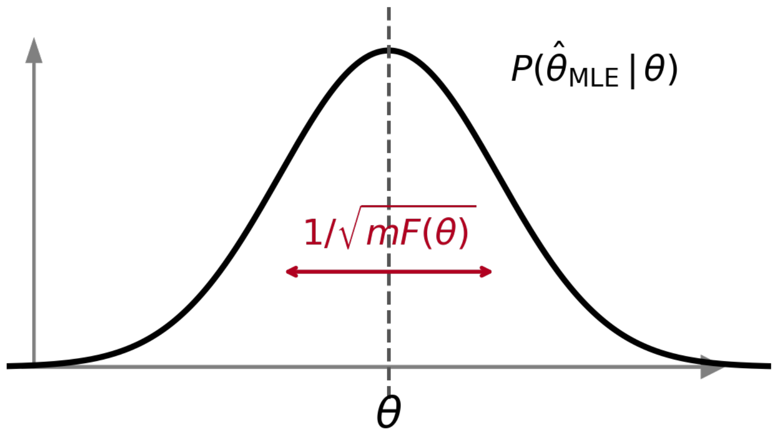
value of the parameter that maximizes the likelihood of the observed events $\mu_1, \ldots, \mu_m$

$$\theta_{\text{MLE}}(\mu_1,\ldots,\mu_m) \equiv \arg\max_{\varphi} P(\mu_1,\ldots,\mu_m \mid \varphi)$$

m is the number of independent measurements

Asymptotically, for many independent repetitions, the estimator distribution becomes Gaussian.

Sampling distribution of the MLE



$$P(\hat{\theta}_{\mathrm{MLE}} \,|\, \theta) \simeq \sqrt{\frac{mF(\theta)}{2\pi}} \exp\left[-\frac{mF(\theta)}{2}(\hat{\theta}_{\mathrm{MLE}} - \theta)^2\right]$$

Centered asymptotically $\mathbb{E}_\theta[\hat{\theta}_{\mathrm{MLE}}] \to \theta$

Variance reaches $\Delta^2 \hat{\theta}_{\mathrm{MLE}} \simeq \frac{1}{mF(\theta)}$

$$P(\theta_{\mathrm{MLE}} \,|\, \theta) = \sqrt{\frac{mF(\theta)}{2\pi}} e^{-\frac{mF(\theta)}{2}(\theta - \theta_{\mathrm{MLE}})^2}$$

Maximum likelihood reaches the Fisher-information limit

$$F(\theta) = \sum_\mu \frac{1}{P(\mu \,|\, \theta)} \left(\frac{dP(\mu \,|\, \theta)}{d\theta} \right)^2$$

QUESTION:

WHAT IS THE HIGHEST SENSITIVITY IN PRECISION MEASUREMENTS ALLOWED BY QUANTUM MECHANICS?

Cramér-Rao bound: Ultimate precision provided by the quantum Fisher information

$$\Delta^2\theta \geq \frac{1}{m}\frac{\Delta^2(\hat{\mu})}{|\langle[\hat{H},\hat{\mu}]\rangle|^2} \geq \frac{1}{mF_{cl}} \geq \frac{1}{mF_Q}$$

m is the number of measurements

M. Frechet, Rev. Inst. Int. Stat. (1943)
C.R. Rao Bull. Calc. Math. Soc. (1945)
H. Cramér *Math. Method of Statistics* (1946)
Helstrom (1967)
Braunstein and Caves (1994)

$$F_{cl} = \sum_{\mu} p_\mu(\theta)\ L_\mu(\theta)^2$$

$$F_Q = Tr[\hat{\rho}\ \hat{L}^2]$$

$$\frac{\partial P_\mu(\theta)}{\partial\theta} = L_\mu(\theta)\ P_\mu(\theta)$$

$$\frac{\partial\hat{\rho}(\theta)}{\partial\theta} = \frac{1}{2}\{\hat{L},\hat{\rho}\}$$

Homework: Proof of the Cramer-Rao lower bound

$$① \quad \sum_{\mu} P(\mu|\theta) \left(\theta^{est}(\mu) - \langle \theta^{est} \rangle\right) = 0$$

$$\left[\langle \theta^{est} \rangle = \sum_{\mu} P(\mu|\theta) \, \theta^{est}(\mu) \right]$$

$$\frac{\partial}{\partial \theta} ① \Rightarrow \sum_{\mu} \left[\frac{\partial P(\mu|\theta)}{\partial \theta} \left(\theta^{est}(\mu) - \langle \theta^{est} \rangle\right) - P(\mu|\theta) \frac{\partial \langle \theta^{est} \rangle}{\partial \theta} \right] = 0$$

$$\left(\sum_{\mu} \frac{1}{\sqrt{P(\mu|\theta)}} \frac{\partial P(\mu|\theta)}{\partial \theta} \sqrt{P(\mu|\theta)} \left(\theta^{est}(\mu) - \langle \theta^{est} \rangle\right) \right)^2 = \left(\frac{\partial \langle \theta^{est} \rangle}{\partial \theta} \right)^2$$

$$\left[\text{Schwartz inequality: } \left| \sum_{\mu} f_\mu g_\mu \right|^2 \le \sum_{\mu} |f_\mu|^2 \sum_{\mu} |g_\mu|^2 \right]$$

$$\Delta^2 \theta \geq \left(\frac{\partial \langle \theta^{est} \rangle}{\partial \theta} \right)^2 \frac{1}{\sum_{\mu} \frac{1}{P(\mu|\theta)} \left(\frac{\partial P(\mu|\theta)}{\partial \theta} \right)^2}$$

$\left(\frac{\partial \langle \theta^{est} \rangle}{\partial \theta} \right)^2$ → = 1 for unbiased estimators

$\sum_{\mu} \frac{1}{P(\mu|\theta)} \left(\frac{\partial P(\mu|\theta)}{\partial \theta} \right)^2$ → Fisher information

Homework:

Examples:

1) $$P(x|\theta) = \frac{1}{\sqrt{2\pi\sigma_\theta^2}} e^{-\frac{(x-\bar{x}_\theta)^2}{2\pi\sigma_\theta^2}}$$

$$\begin{cases} \bar{x} = x(\theta) \\ \sigma = \sigma(\theta) \end{cases}$$

Prove that:

$$F = \int dx \frac{1}{P(x|\theta)} \left(\frac{\partial P(x|\theta)}{\partial \theta}\right)^2 =$$

$$= \frac{1}{\sigma^2}\left(\frac{\partial \bar{x}}{\partial \theta}\right)^2 + \frac{2}{\sigma^4}\left(\frac{\partial \sigma}{\partial \theta}\right)^2$$

Homework: A lower bound for the Fisher information

$$\bar{m} = \sum_m P(m|\theta)\, m$$

$$\Delta^2 m = \sum_m P(m|\theta)\, (m-\bar{m})^2$$

$$\frac{\partial \bar{m}}{\partial \theta} = \sum_m m \frac{\partial P(m|\theta)}{\partial \theta} = \sum_m (m-\bar{m}) \frac{\partial P(m|\theta)}{\partial \theta} \qquad \left[\sum_m \frac{\partial P(m|\theta)}{\partial \theta} = 0\right]$$

$$\left(\frac{\partial \bar{m}}{\partial \theta}\right)^2 \le \sum_m (m-\bar{m})^2 P(m|\theta) \sum_m \frac{1}{P(m|\theta)} \left(\frac{\partial P(m|\theta)}{\partial \theta}\right)^2$$

$$\frac{1}{\Delta^2 \bar{m}} \left(\frac{\partial \bar{m}}{\partial \theta}\right)^2 \le F$$

$$\frac{1}{\Delta^2 \hat{M}} \frac{\partial \langle \hat{M} \rangle}{\partial \theta} \le F \qquad ; \quad \hat{M} = \sum_m c_m |m\rangle\langle m|$$

$$\Delta^2 \theta_{est} \ge \frac{\Delta^2 \langle \hat{M} \rangle}{\left|\frac{\partial \langle \hat{M} \rangle}{\partial \theta}\right|^2} \ge \frac{1}{F}$$

Spin squeezing $\quad \hat{M} \equiv \hat{J}_z$

$$\frac{\partial \langle \hat{J}_z \rangle}{\partial \theta} = i[\hat{J}_y, \hat{J}_z] = \hat{J}_x$$

$\Downarrow$

$$\frac{\Delta^2 \hat{J}_z}{|\langle \hat{J}_x \rangle|^2} \ge \frac{1}{N}$$

Theory of Entanglement-enhanced phase estimation

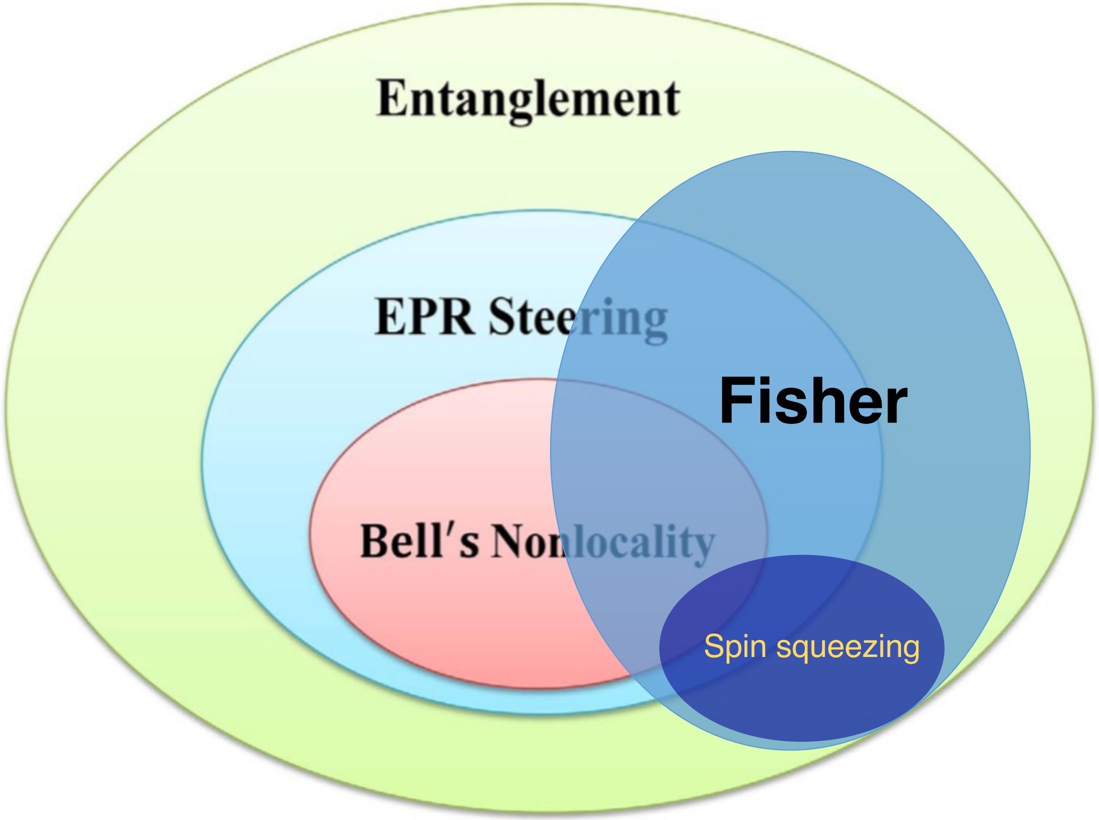
Metrological useful entanglement

PARAMETER ESTIMATION SENSITIVITY IS PROVIDED BY THE FISHER INFORMATION (1945)

$$\Delta^2\theta_{est} = \frac{1}{F}$$

THE FISHER INFORMATION RECOGNIZES MULTIPARTITE ENTANGLEMENT IN A SUBCLASS OF ALL ENTANGLED STATES (2009)

$$F > N$$



THE FISHER INFO CRITERIA CERTIFIES METROLOGICALLY USEFUL ENTANGLEMENT (I.E. NECESSARY AND SUFFICIENT FOR QUANTUM-ENHANCED PARAMETER ESTIMATION)

This is one of a few cases where it is possible to establish a direct relation between quantum advantage and a class of entangled states for a specific quantum technology

Quantum Interferometry: central message

Given a state $\hat{\rho}$ of N qbits
a generator of a unitary local transformation $\hat{H}$
and a measurement observable $\hat{M}$

If $F(\hat{\rho}, \hat{H}, \hat{M}) > N \;\rightarrow\; \hat{\rho}$ is particle entangled (sufficient)

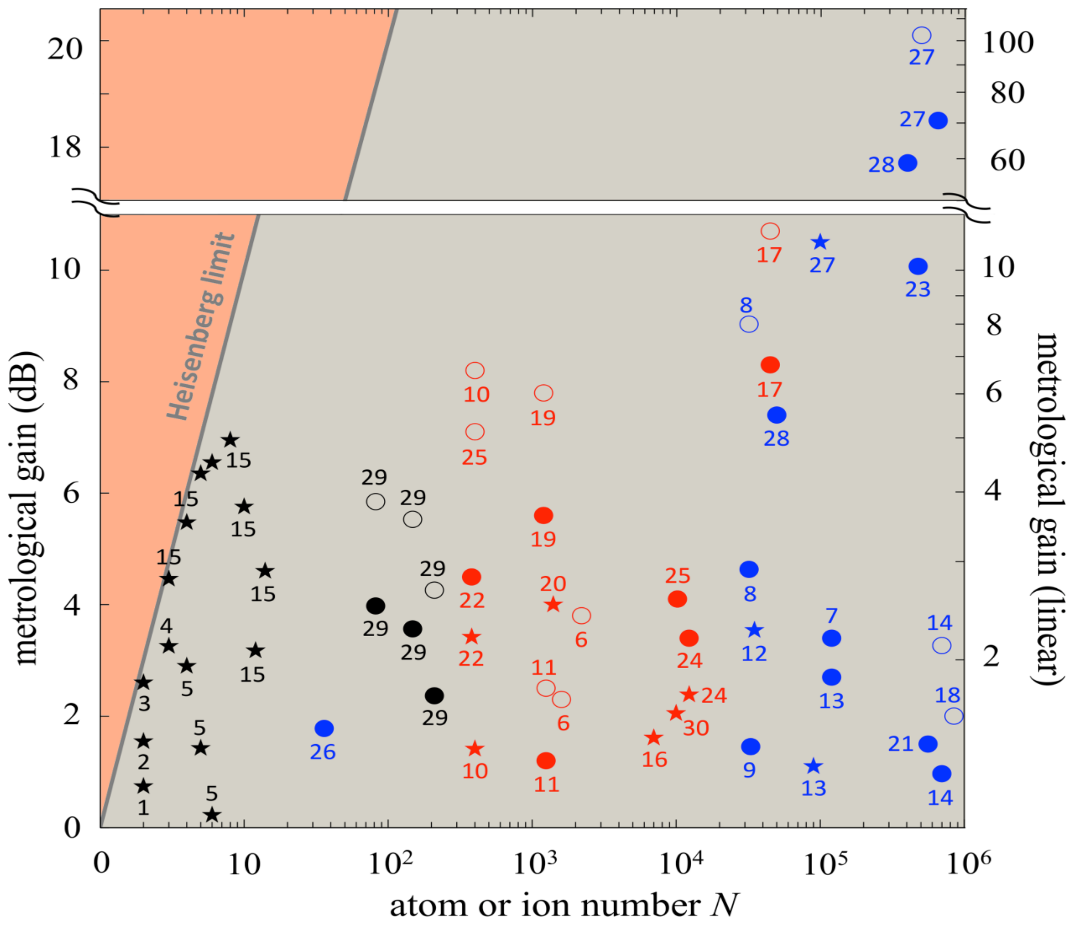
$$F = \sum_{\mu} P(\mu|\theta)[\partial_\theta \ln P(\mu|\theta)]^2$$

the interferometer $\hat{\rho}(\theta) = e^{-i\hat{H}\theta} \; \hat{\rho} \; e^{i\hat{H}\theta}$
with the observable $\hat{M}$
provides sub shot-noise sensitivity (necessary & sufficient)

$$\Delta\theta \underset{m \gg 1}{=} \frac{1}{\sqrt{m \, F}} \quad \Rightarrow \quad \Delta\theta = \frac{1}{\sqrt{N_{tot}}} \qquad \Delta\theta = \frac{1}{N_{tot}}$$

from shot noise to Heisenberg limit

Searching for sub shot-noise up to Heisenberg limit sensitivities



TRAPPED IONS

[1] Sackett *et al.*, 2000 NIST
[2] Meyer *et al.*, 2001 NIST
[3] Leibfried *et al.*, 2003 NIST
[4] Leibfried *et al.*, 2004 NIST
[5] Leibfried *et al.*, 2005 NIST
[15] Monz *et al.*, 2011 Innsbruck
[29] Bohnet *et al.*, 2015 NIST

BOSE-EINSTEIN CONDENSATES

[6] Estève *et al.*, 2008 Heidelberg
[10] Gross *et al.*, 2010 Heidelberg
[11] Riedel *et al.*, 2010 Besel
[16] Lücke *et al.*, 2011 Hannover
[17] Hamley *et al.*, 2012 GeorgiaTech
[19] Berrada *et al.*, 2013 Wien
[20] Ockeloen *et al.*, 2013 Basel
[22] Strobel *et al.*, 2014 Heidelberg
[24] Muessel *et al.*, 2014 Heidelberg
[25] Muessel *et al.*, 2015 Heidelberg
[30] Kruse *et al.*, 2016 Hannover

COLD THERMAL ATOMS

[7] Appel *et al.*, 2009 Copenhagen
[8] Leroux *et al.*, 2010a MIT
[9] Schleier-Smith *et al.*, 2010 MIT
[12] Leroux *et al.*, 2010b MIT
[13] Louchet-Chauvet *et al.*, 2010 Copenhagen
[14] Chen *et al.*, 2011 JILA
[18] Sewell *et al.*, 2012 ICFO
[21] Sewell *et al.*, 2014 ICFO
[23] Bohnet *et al.*, 2014 JILA
[26] Barontini *et al.*, 2015 ENS
[27] Hosten *et al.*, 2016a Stanford
[28] Cox *et al.*, 2016 JILA

Pezzè, Smerzi, Oberthaler, Schmied & Treutlein RMP 2018

How to create Entanglement?

one-axis twisting

$$|\psi_{\mathrm{OAT}}(t)\rangle = e^{-i\chi t \hat{J}_z^2}|\mathrm{CSS}\rangle_x$$

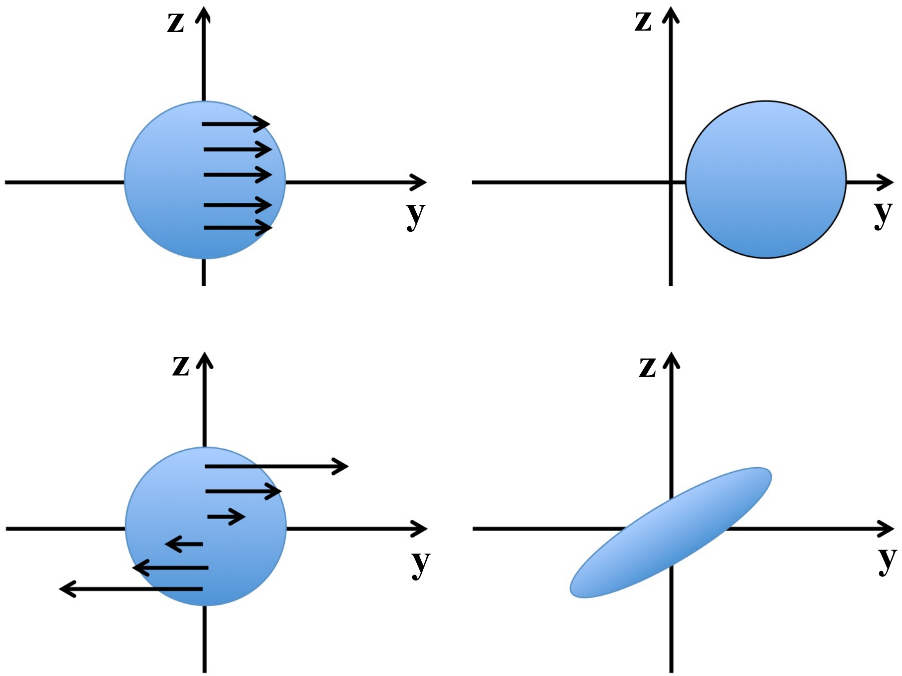
Kitagawa & Ueda, PRA 47, 5138 (1993)

rigid rotation

$e^{-i\chi t \hat{J}_z}$

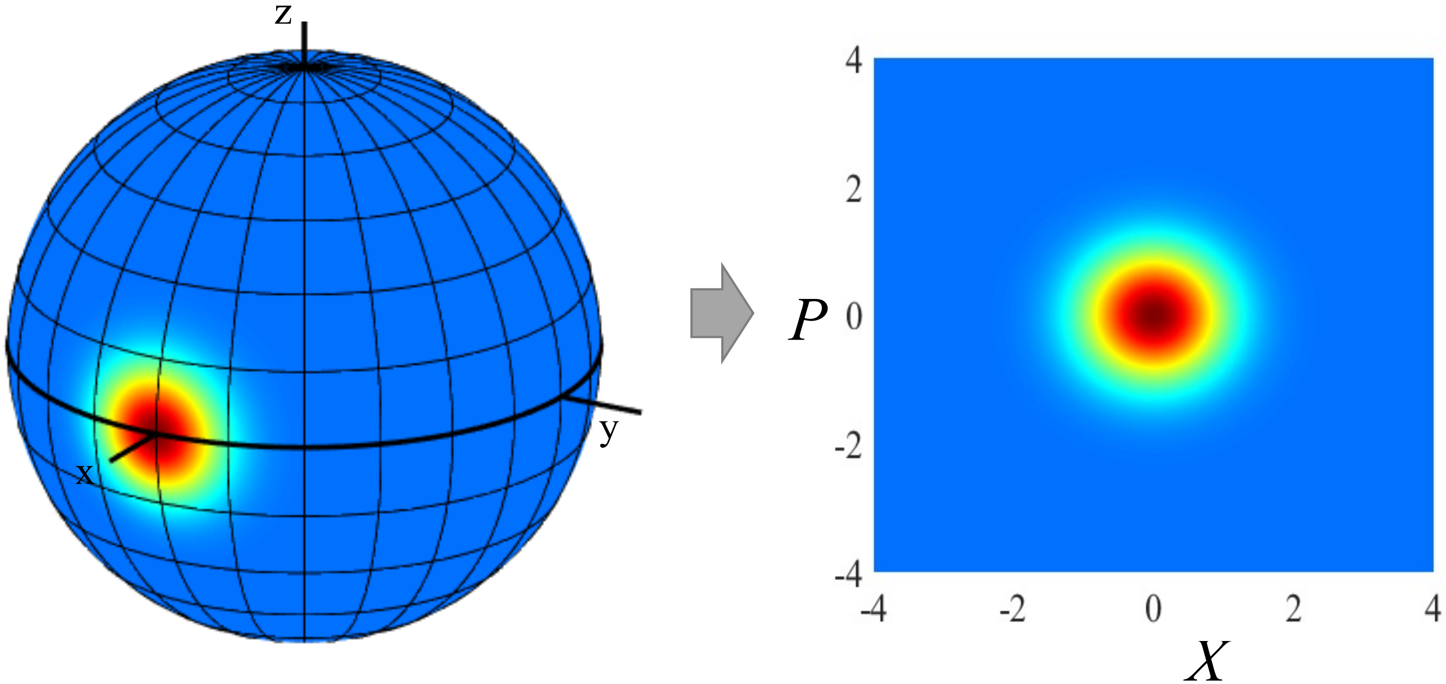
one-axis twisting

$e^{-i\chi t \hat{J}_z^2} \approx e^{-i(\chi t J_z)\hat{J}_z}$



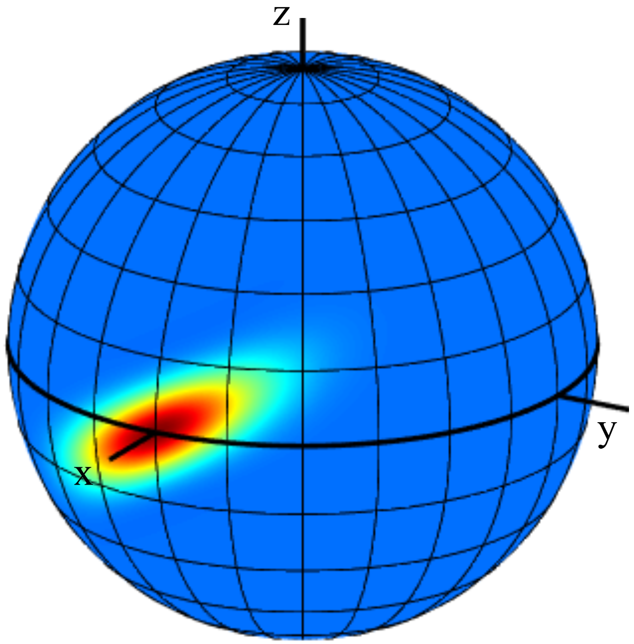
Twisting dynamics can be understood *heuristically* as a “rotation” of a J_z-dependent angle

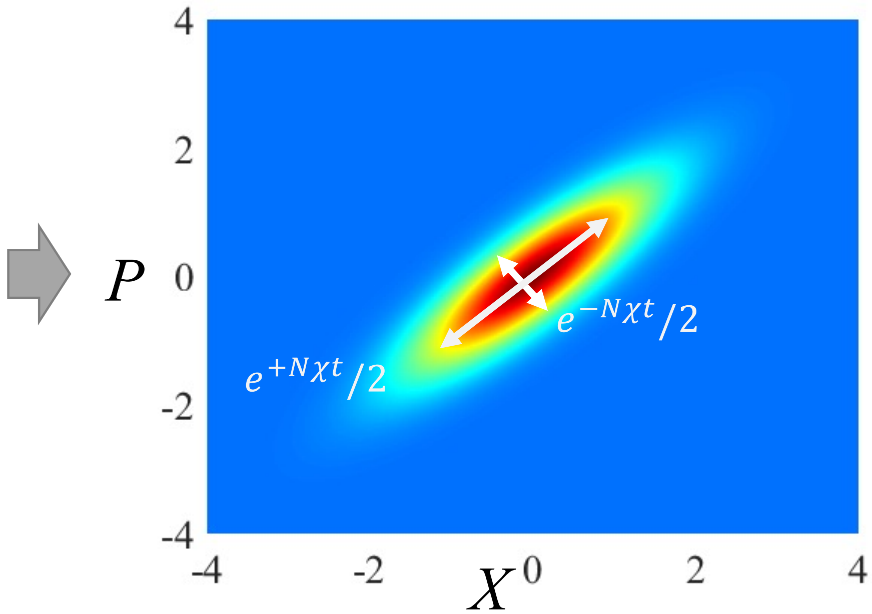
one-axis twisting: short time dynamics



$$|\psi_{\mathrm{OAT}}(t)\rangle = e^{-i\chi t \hat{J}_z^2} |\mathrm{CSS}\rangle_x$$

one-axis twisting: short time dynamics





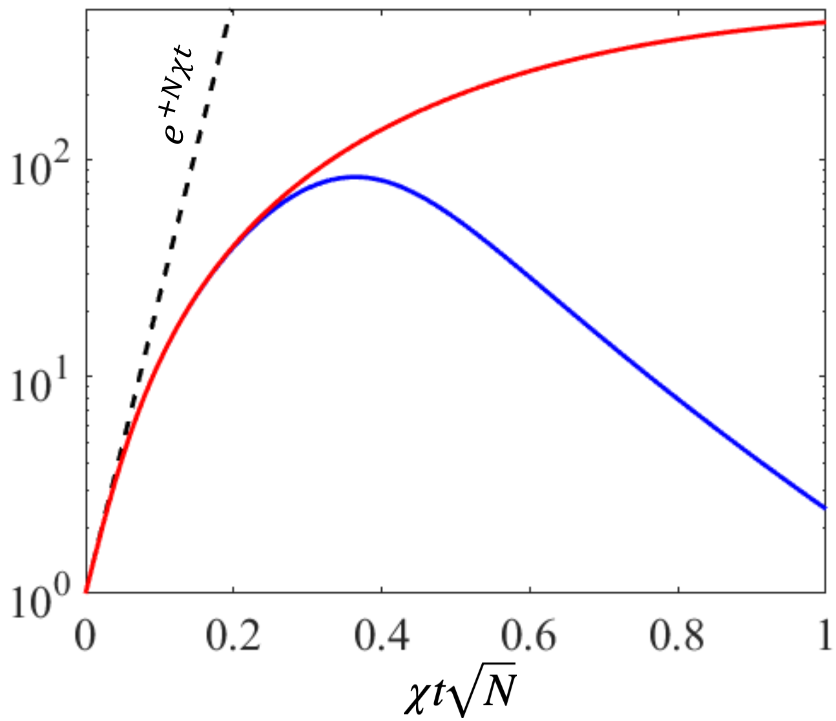
$$\xi_R^2 \approx e^{-N\chi t}$$

exponential decrease of the squeezing parameter

$$F_Q / N \approx e^{+N\chi t}$$

exponential increase of the quantum Fisher information

one-axis twisting: short time dynamics



— $1/\xi_R^2$

$$\xi_R^2 = \frac{4 + (N-1)\left(\alpha(\tau) - \sqrt{\alpha^2(\tau) + \beta^2(\tau)}\right)}{4\cos^{2N-2}(\tau)}$$

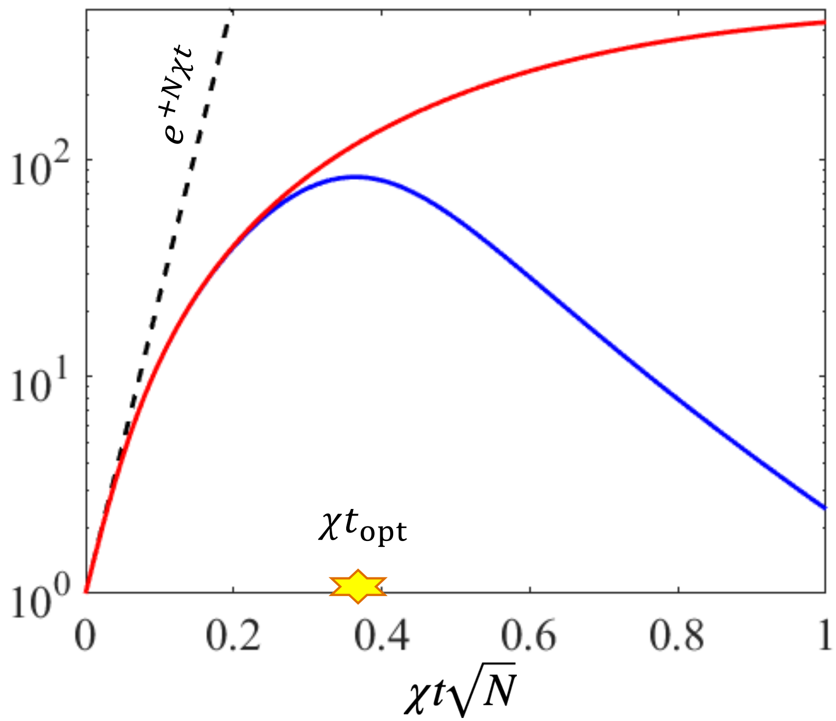
$$\alpha(\tau) = 1 - \cos^{N-2}(2\tau)$$

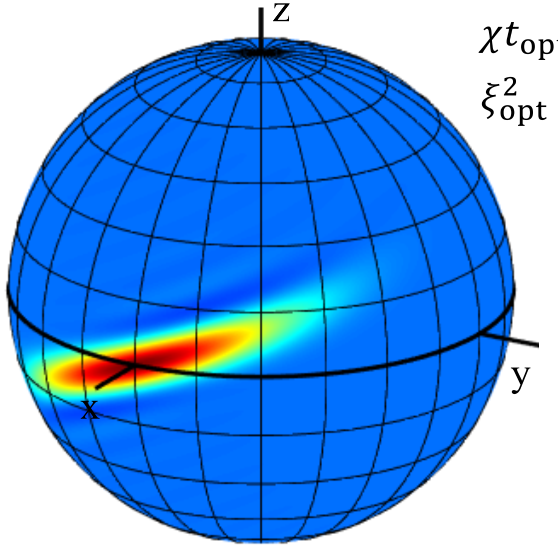
$$\beta(\tau) = 4\sin(\tau)\cos^{N-2}(\tau)$$

— F_Q/N

$$F_Q = N + \frac{N(N-1)}{4}\left(\alpha(\tau) + \sqrt{\alpha^2(\tau) + \beta^2(\tau)}\right)$$

one-axis twisting: short time dynamics



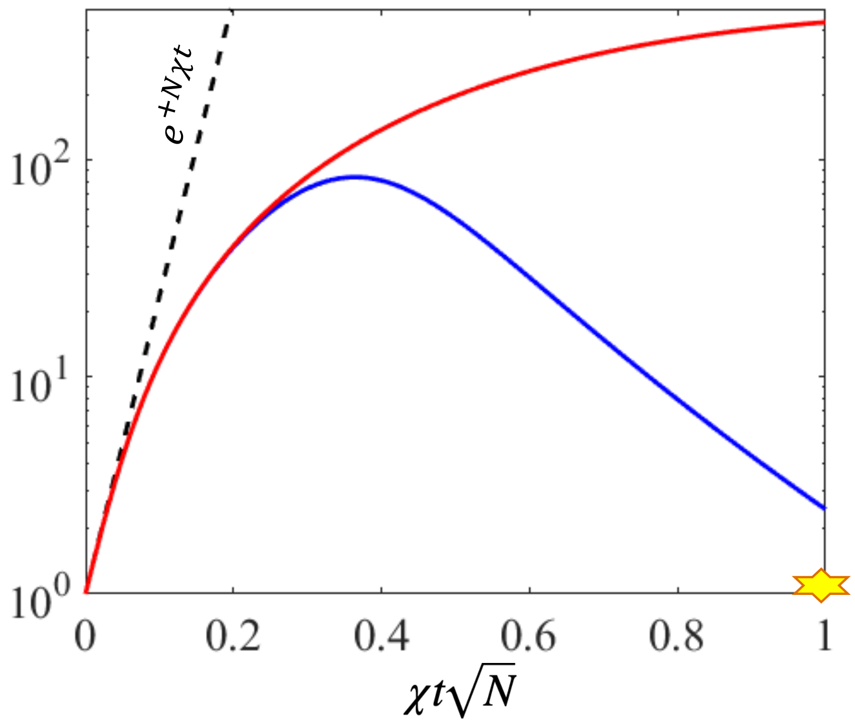


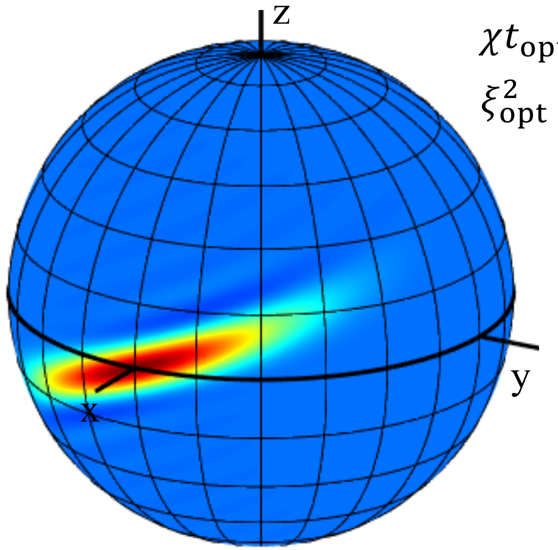
$\chi t_{\text{opt}} \sim N^{-2/3}$

$\xi^2_{\text{opt}} \sim N^{-2/3}$

optimal squeezing point due to the bending of the state in the Bloch sphere ($\langle \hat{J}_z \rangle \ll N/2$, $\Delta^2 \hat{J}_z \approx N^2$)

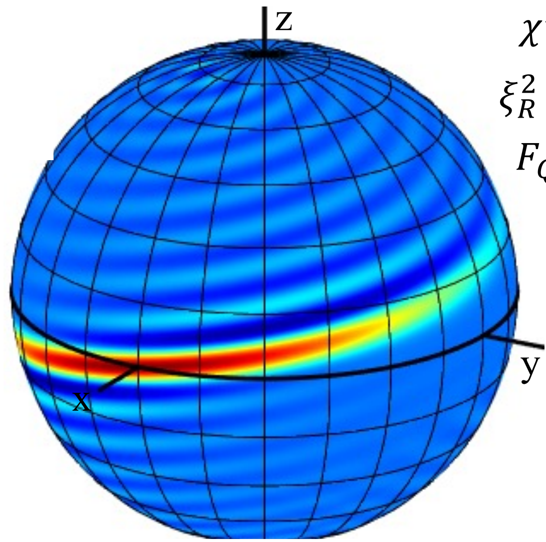
one-axis twisting: short time dynamics





$\chi t_{\mathrm{opt}} \sim N^{-2/3}$

$\xi^2_{\mathrm{opt}} \sim N^{-2/3}$



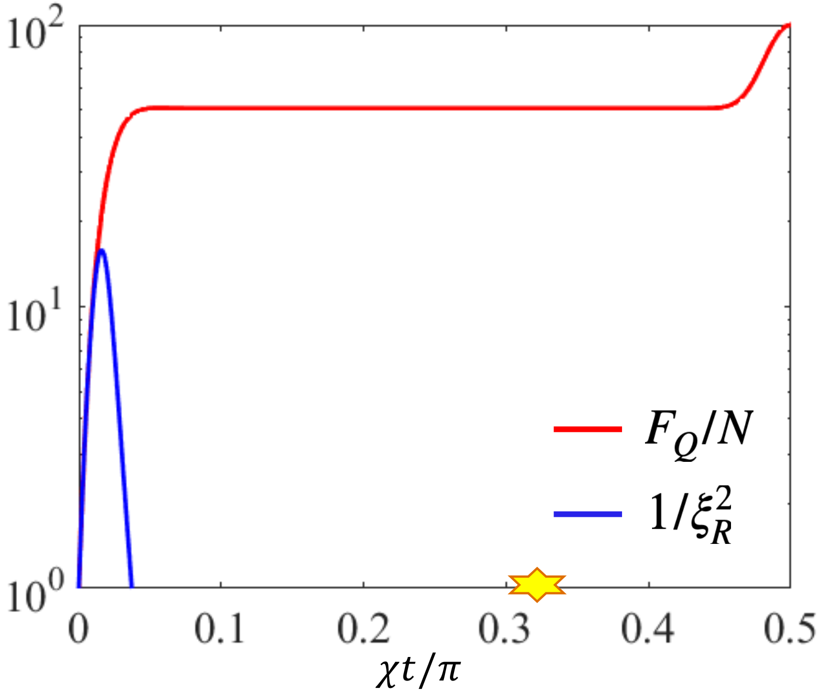
$\chi t \sim N^{-1/2}$

$\xi^2_R \sim 1$

$F_Q \sim N^2/2$

the state is not spin squeezed but still usefully entangled: $F_Q > N$

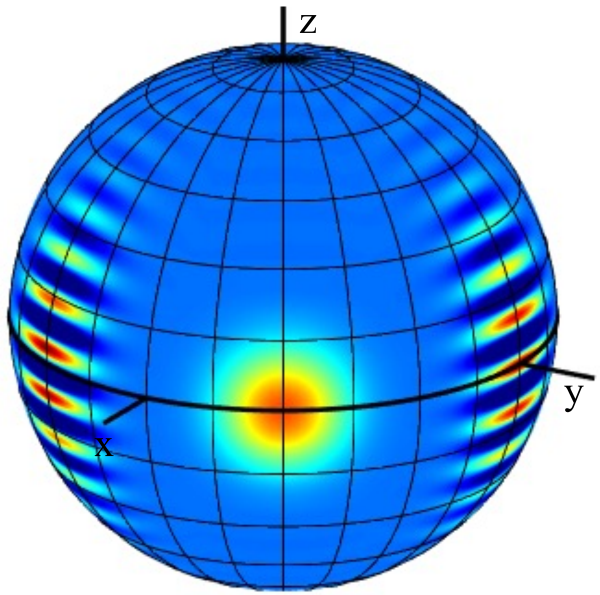
one-axis twisting: long-time dynamics



precisely at time $\chi t = \pi/n$ the state evolves into a coherent superposition of $n \sim \sqrt{N}/2 \in \mathbb{N}$ CSSs evenly distributed

Agarwal et al., PRA 56, 2249 (1997)

es. $\chi t = \pi/3$



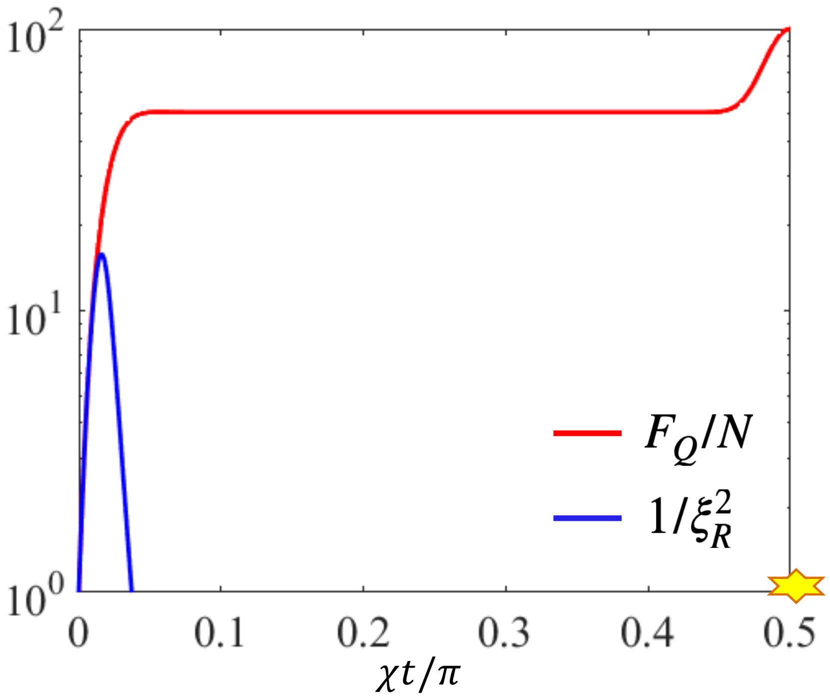
long plateau where F_Q is constant:

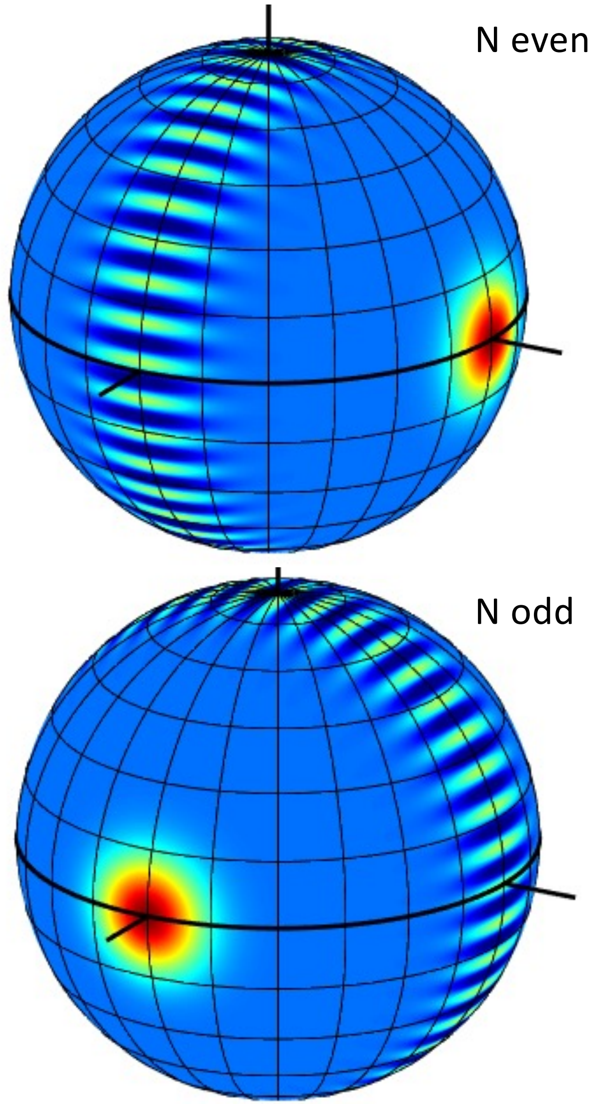
$1/\sqrt{N} \lesssim \chi t \lesssim \pi/2 - 1/\sqrt{N}$

$F_Q = N^2/2$

Pezzè and Smerzi, PRL 102, 100401 (2009)

one-axis twisting: long-time dynamics





At $\chi t = \pi/2$: perfect GHZ (or NOON) state
(along z for N even– along x for N odd)

$$F_Q = N^2$$

Mølmer and Sørensen PRL 82, 1835 (1999)

The heretic approach:

Bayesian phase estimation

WHAT IS PROBABILITY ?

Frequentist interpretation. Objective definition (Fisher,…):

Probability exists outside the human mind and can indeed be measured.
Based on law of large numbers: frequencies fluctuations decrease with the number of trials. (Notice: this is the standard interpretation in QM).
The maximum likelihood approach is frequentist.

$$P = \lim_{N \to \infty} \frac{n}{N}$$

n favorable outcomes out of N possible outcomes

WHAT IS PROBABILITY ?

Bayesian interpretation. Subjective definition (Bayes, de Finetti...):

Probability exists only in the human mind.

Example:
A vessel contains N balls. Each ball can be blue or red.
Without looking inside the vessel, you pick up a ball and get a blue one.

Question: How much money you'll bet that if you try again you get another blue ball?
Same question: what is the probability that the vessel contains n blue balls?

Answers:

Frequentist: the question does not make sense, I can answer only after picking up several balls, with # of balls >>1, and calculate frequencies.

Bayesian: the probability is proportional to the money you'll bet that the vessel contains n blue balls

Probability is a subjective degree of belief based on available data and "rational" opinions.

The Bayesian approach is typical in real life (insurance, horse races,…)
As humans, we have an intuitive understanding of probability and take decisions with a Bayesian line of reasoning.

BAYES RULE

$$P(\mu,\theta) = P(\mu|\theta)P(\theta) = P(\theta|\mu)P(\mu) = P(\theta,\mu)$$

$$P_{post}(\theta|\mu) = P_{lik}(\mu|\theta)\ P_{prior}(\theta)/P_{marg}(\mu)$$

Likelihood function.
This is known.

The marginal probability is fixed by the normalization condition

Posterior probability:
It provides a "degree of belief" about the true value of the phase shift given the measured data.

Prior function: it expresses the prior state of knowledge or belief about the true value of θ

Example :

A vessel contains N balls, that can be either blue or red.

Without looking inside the vessel, you pick up a ball and get it blue.

Question:

After picking one blue ball, how much would you bet that the next ball you pick will also be blue?

Homework:

A vessel contains N balls, that can be either blue or red.

Without looking inside the vessel, you pick up a ball and get it blue.

Question:

After picking one blue ball, how much would you bet that the next ball you pick will also be blue?

Bayesian approach

What is the prior (before picking up the blue ball) probability to have k blue balls in the vessel?

$$P(k) = \frac{1}{N+1}$$

assume maximum ignorance

Bayes rule

$$P(k|1) = \frac{P(1|k)\ P(k)}{P(1)} = \frac{2k}{N(N+1)}$$

Probability / degree of belief that the vessel contains k blue balls given one blue ball was extracted

Picking up several balls, Bayesian and frequentist probabilities converge to same result.

Solution:

Assume a vessel contains N balls, of which k are blue, but k is unknown. We use a uniform prior:

$$P(k) = \frac{1}{N+1} \quad \text{for } k = 0, 1, \ldots, N$$

Suppose the first ball drawn is blue. We want to compute the conditional probability:

$$P(B_2 = 1 \mid B_1 = 1)$$

Using Bayesian inference:

$$P(B_2 = 1 \mid B_1 = 1) = \sum_{k=1}^{N} P(B_2 = 1 \mid B_1 = 1, k) \cdot P(k \mid B_1 = 1)$$

Where:

$$P(B_2 = 1 \mid B_1 = 1, k) = \frac{k-1}{N-1} \quad \text{and} \quad P(k \mid B_1 = 1) = \frac{P(B_1 = 1 \mid k) \cdot P(k)}{P(B_1 = 1)} = \frac{\frac{k}{N} \cdot \frac{1}{N+1}}{P(B_1 = 1)}$$

Compute the marginal:

$$P(B_1 = 1) = \sum_{k=0}^{N} \frac{k}{N} \cdot \frac{1}{N+1} = \frac{1}{N(N+1)} \sum_{k=0}^{N} k = \frac{1}{N(N+1)} \cdot \frac{N(N+1)}{2} = \frac{1}{2}$$

Thus:

$$P(k \mid B_1 = 1) = \frac{\frac{k}{N} \cdot \frac{1}{N+1}}{1/2} = \frac{2k}{N(N+1)}$$

Now compute:

$$P(B_2 = 1 \mid B_1 = 1) = \sum_{k=1}^{N} \frac{k-1}{N-1} \cdot \frac{2k}{N(N+1)} = \frac{2}{N(N+1)(N-1)} \sum_{k=1}^{N} k(k-1)$$

Evaluate the sum:

$$\sum_{k=1}^{N} k(k-1) = \sum_{k=1}^{N} (k^2 - k) = \frac{N(N+1)(2N+1)}{6} - \frac{N(N+1)}{2} = \frac{N(N+1)}{6}(2N+1-3) = \frac{N(N+1)(2N-2)}{6}$$

Therefore:

$$P(B_2 = 1 \mid B_1 = 1) = \frac{2}{N(N+1)(N-1)} \cdot \frac{2N(N+1)(N-1)}{6} = \frac{4}{6} = \boxed{\frac{2}{3}}$$

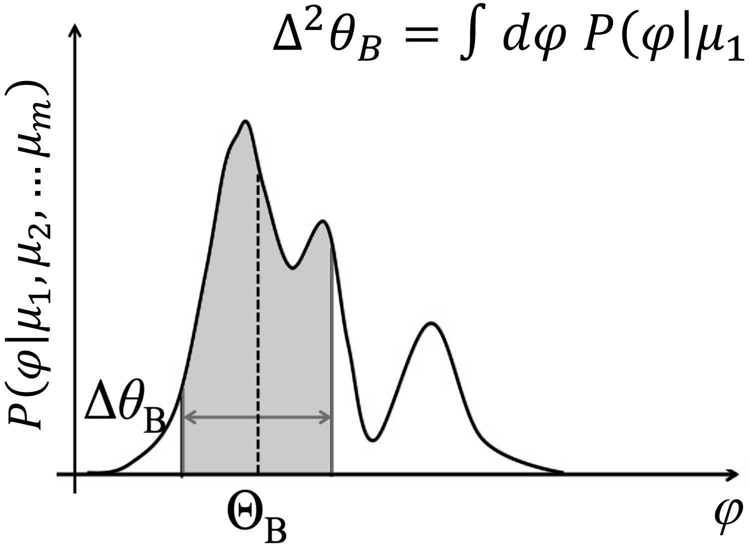
Bayesian phase estimation

frequentist approach: the uncertainty is calculated from histograms
Bayesian setting: the uncertainty is calculated from the Bayesian distribution

$$P(\varphi|\mu_1,\mu_2,\dots\mu_m) = \frac{P(\mu_1,\mu_2,\dots\mu_m|\varphi)P_{\text{prior}}(\varphi)}{P(\mu_1,\mu_2,\dots\mu_m)}$$

degree of belief that $\varphi = \theta$ given the observed results

$$\Delta^2\theta_B = \int d\varphi \, P(\varphi|\mu_1,\dots)(\varphi - \Theta_B)^2$$



Nolan, Smerzi & LP, NPJQI (2020)

the Cramer-Rao bound *does not* apply
but consistent asymptotic properties
in the central limit ($m \gg 1$)

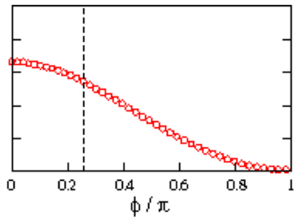
$$P(\varphi|\mu_1,\dots\mu_m) = \sqrt{\frac{mF(\theta)}{2\pi}} e^{-\frac{mF(\theta)}{2}(\varphi-\theta)^2}$$

$$\Delta^2\theta_B = 1/mF(\theta)$$

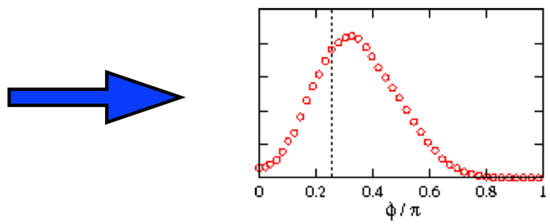
Experiment: i*)* collect measurements (number of particles at the output ports) and
ii) associate the Bayesian phase distribution to each detection

e.g.:

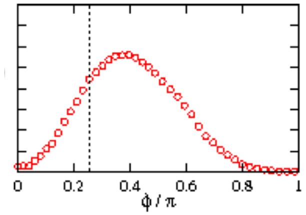
1st measurement



multiply the distributions:

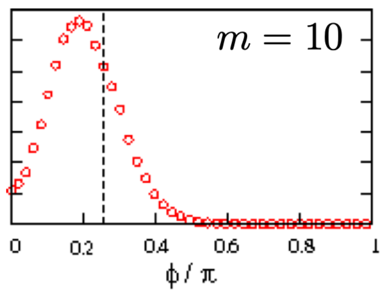


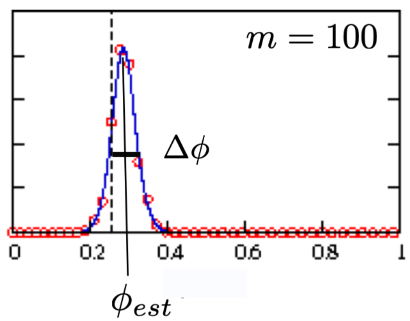
2nd measurement



after m measurements

$$P_{exp}(\phi|\{N_C^{(i)}, N_D^{(i)}\}_{i=1....m}) \sim \prod_{i=1}^{m} P_{exp}(\phi|N_C^{(i)}, N_D^{(i)})$$





Conclusion

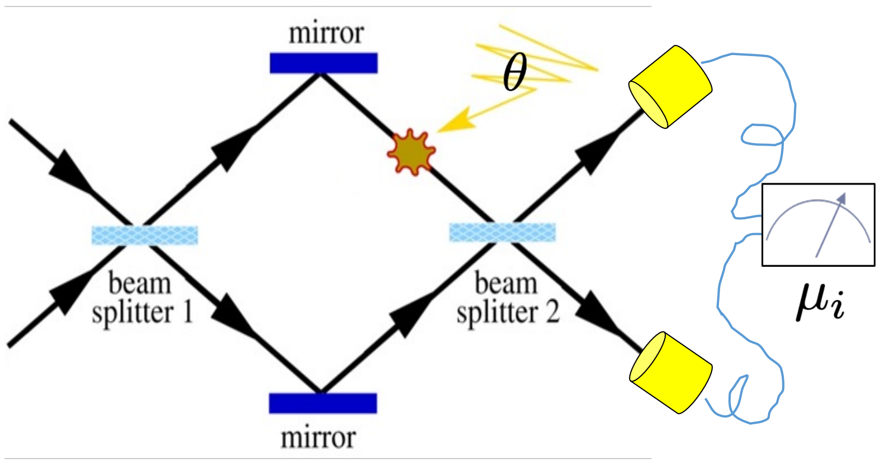
If there is no uncertainty, it cannot be true
R. P. Feynman

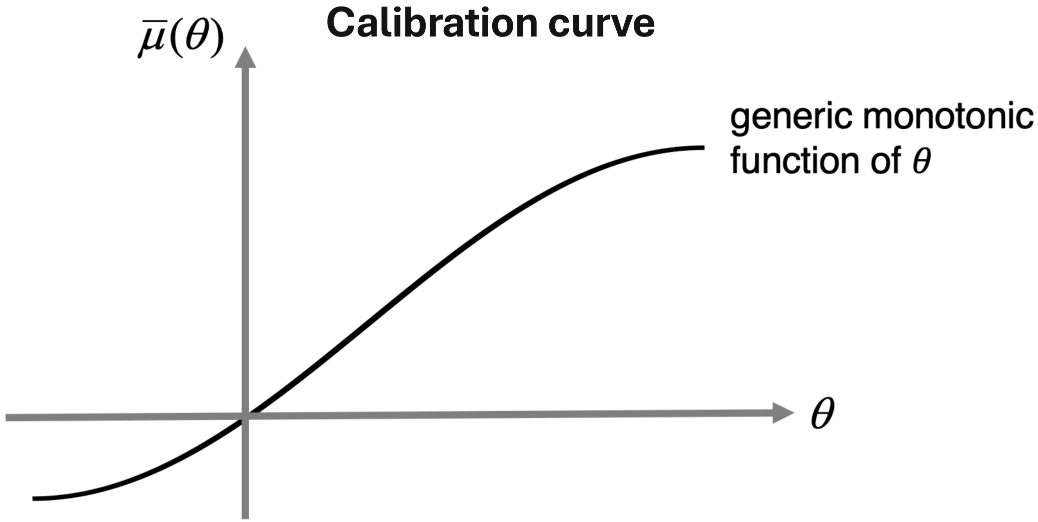
APPENDIX

Method of moments: use the mean signal

The interferometer maps the unknown phase into the average of a measured observable.

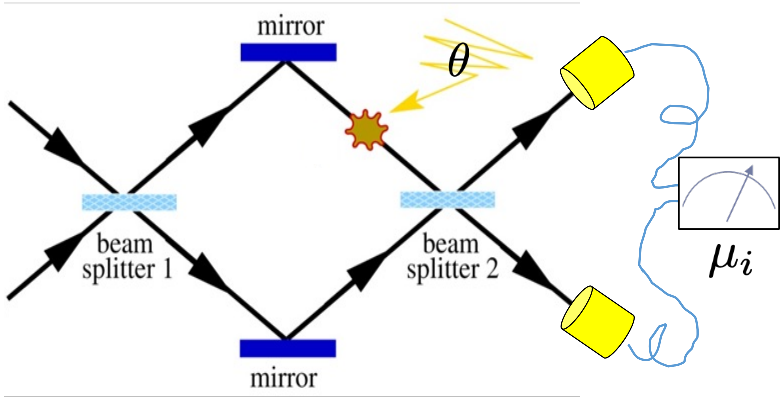
Key condition: use a local region where the calibration curve is invertible.





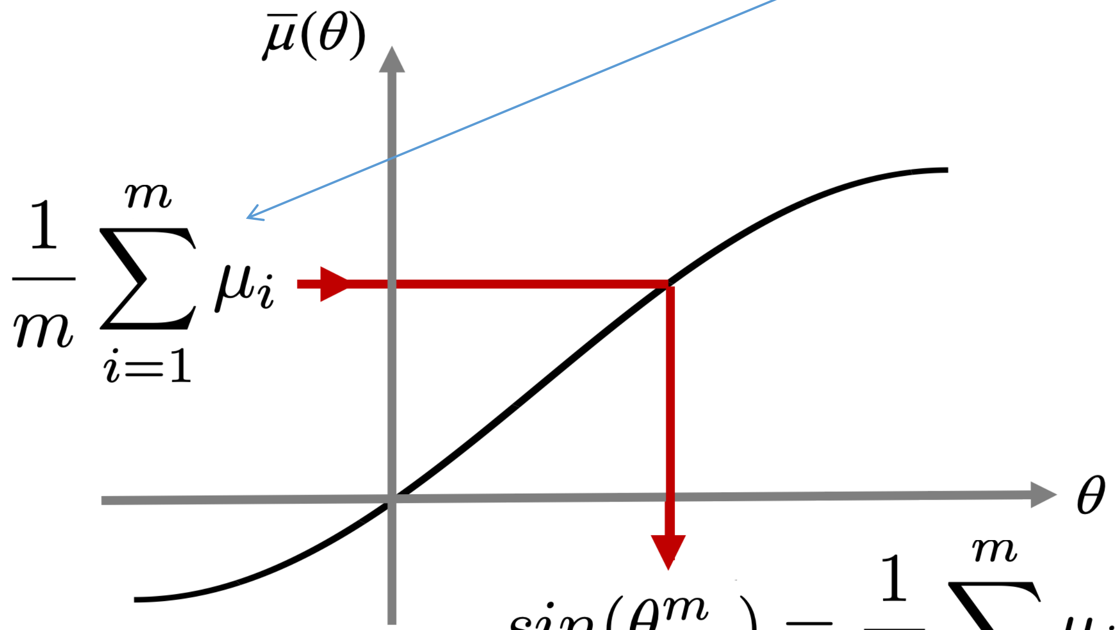
m is the number of independent measurements

$$E.g.\ \bar{\mu} = \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \mu_i = sin(\theta)$$



Repeat the interferometric measurement *m* times and calculate the outcome frequency

The estimated value of the phase is obtained by replacing average with frequency and inverting the measured mean



$$sin(\theta^m_{est}) = \frac{1}{m}\sum_{i=1}^{m}\mu_i$$

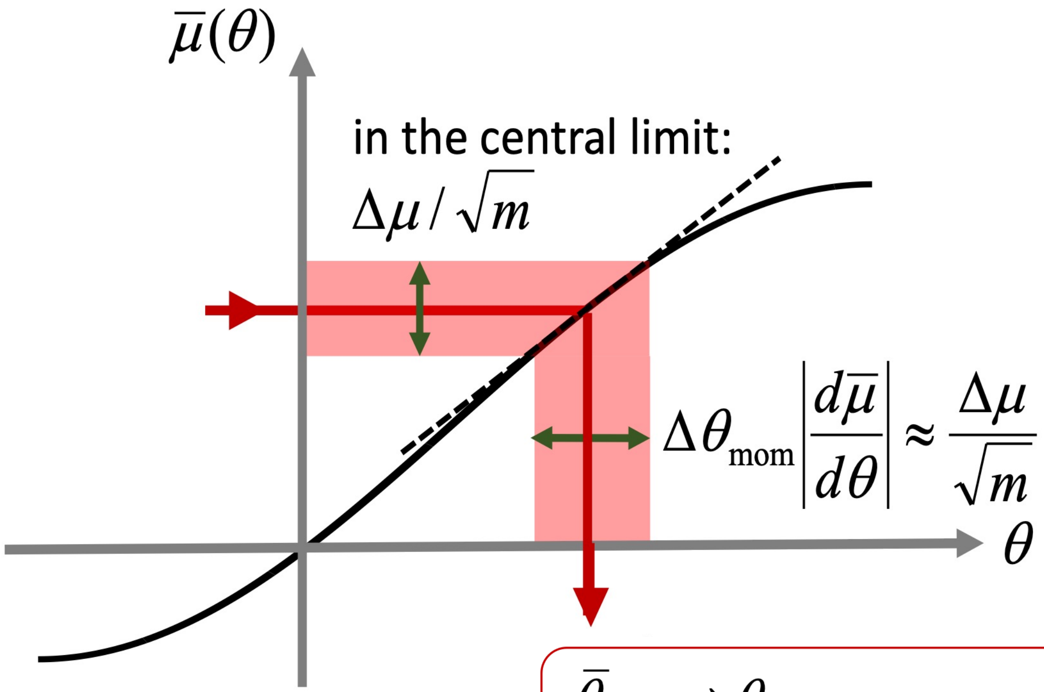
m is the number of independent measurements

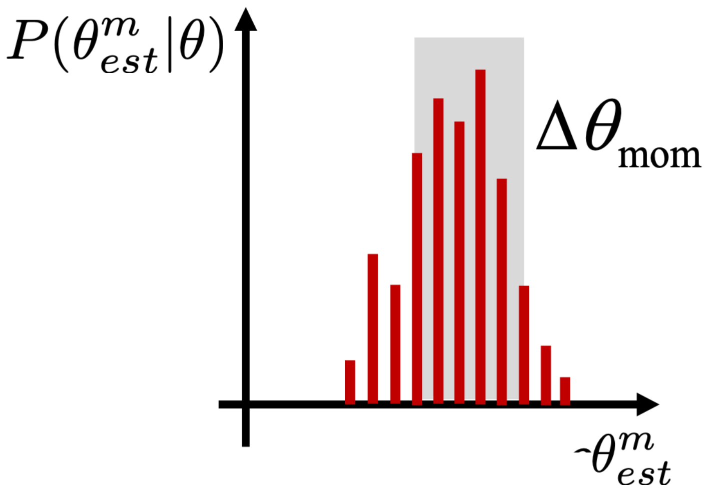
Notice: The estimate is a random variable !

Error propagation converts signal noise into phase noise

Repeat the protocol several times and plot histograms

For large m, the sample mean becomes Gaussian by the central-limit theorem.





m is the number of independent measurements

$$\bar{\theta}_{\mathrm{mom}} \to \theta$$

$$\Delta^2\theta_{\mathrm{mom}} = \frac{\Delta^2\mu}{m(d\bar{\mu}/d\theta)^2}$$

valid in the central limit (sufficiently large m)