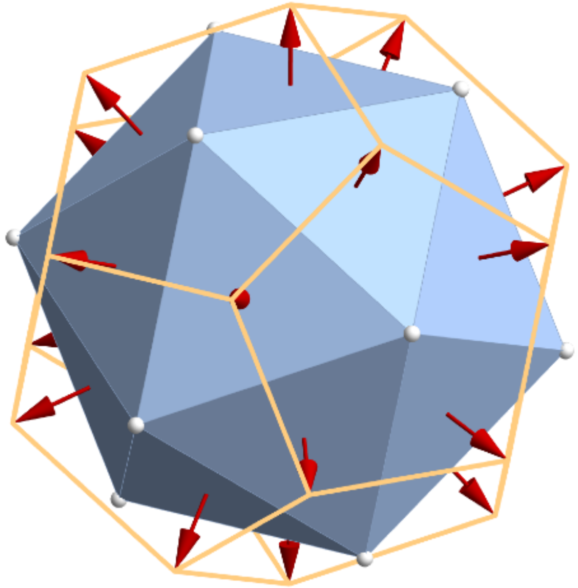
universität innsbruck

Programmable Quantum Sensors and Quantum Compasses



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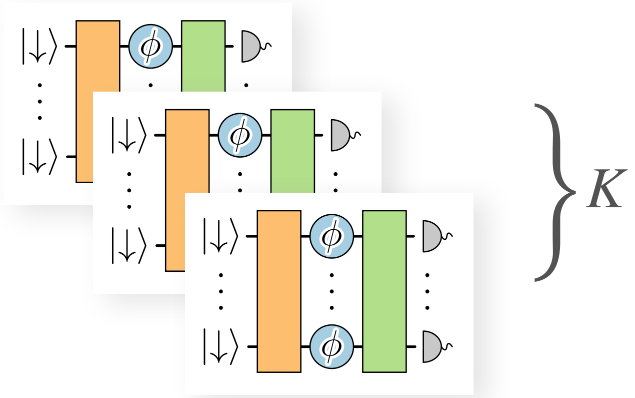
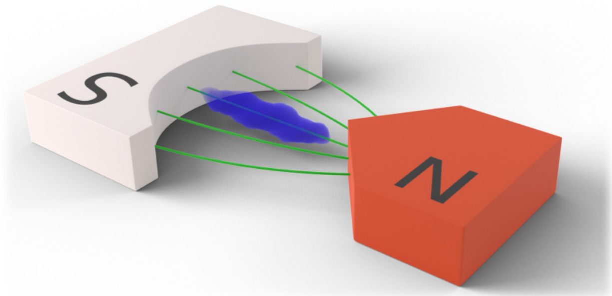
Multiparameter metrology — Many-Repetition Scenario

- Encode a vector of d parameters
  $\boldsymbol{\phi} = \{\phi_1, \phi_2, \ldots, \phi_d\}$
  into a quantum state
- Estimate parameters using K independent measurements



Paradigmatic example:



Vector field sensing

$$\hat{H} = B_x \hat{J}_x + B_y \hat{J}_y + B_z \hat{J}_z, \qquad [\hat{J}_x, \hat{J}_y] = i\hat{J}_z$$

Fundamental Distinction from Single-Parameter Metrology:
Incompatibility of optimal measurements

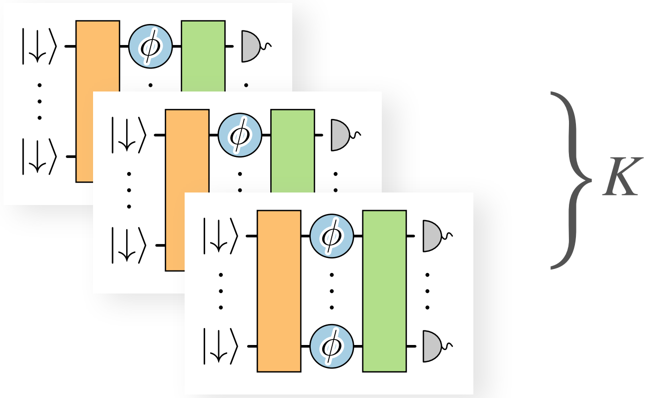
- Fundamental precision bounds are given by the Fisher information framework.
- What is the optimal quantum sensor achieving these bounds?
- How to build it?

arXiv:2404.14194
DV, A. Shankar, R. Kaubruegger, P. Zoller

Fundamental Precision Limits

Cramer-Rao bound and Fisher information approach

The aim is to estimate deviation of parameters from $\boldsymbol{\phi}_0$ using K independent measurements:



$$|\psi_{\boldsymbol{\phi}}\rangle = U(\boldsymbol{\phi})\,|\psi_{\text{in}}\rangle$$

$$\boldsymbol{\mu} = (\mu_1, \ldots, \mu_K)^T$$

vector of K measurement outcomes

Figure of merit is the mean squared error:

$$\text{MSE}(\phi) = \sum_{\boldsymbol{\mu}} (\phi - \xi_{\boldsymbol{\mu}})^2\, p(\boldsymbol{\mu}\,|\,\phi)$$

Fisher Information Matrix
sets bounds on the achievable MSE

Conditional probability (Likelihood):

$$p(\boldsymbol{\mu}\,|\,\phi) = \prod_{k=1}^{K} p(\mu_k\,|\,\phi),$$

$$p(\mu\,|\,\phi) = \text{Tr}\{M_\mu\,|\psi_{\boldsymbol{\phi}}\rangle\langle\psi_{\boldsymbol{\phi}}\,|\,\}, \quad \sum_\mu M_\mu = \mathbb{1}, \quad M_\mu \geq 0$$

Matteo G. A. Paris, Int. J. Quant. Inf. 7, 125 (2009)
Jing Liu et al 2020 J. Phys. A: Math. Theor. 53 023001

Cramer-Rao bound and Fisher information approach

- We impose unbiasedness constraint on estimators: $\langle \boldsymbol{\xi}_m \rangle_{p_{\boldsymbol{\phi}}(m)} = \boldsymbol{\phi}$
- Then the MSE for K measurements is limited by the Fisher information $F = \sum_{\mu} \frac{\nabla p(\mu\,|\,\phi)\nabla^T p(\mu\,|\,\phi)}{p(\mu\,|\,\phi)}$

$$K \times \mathrm{MSE}(\boldsymbol{\phi}) \geq \mathrm{Tr}\{F^{-1}\} \quad \text{the CRB}$$
$$\geq \mathrm{Tr}\{F_Q^{-1}\} \quad \text{the QCRB,}$$
$$\geq \Delta_{\mathrm{HL}} \sim d^2/N^2 \quad \text{the Heisenberg limit}$$

Depends on state and measurement

$$F_Q = \max_{\{M_\mu\}} F$$

- The CRB is saturated by the Maximum Likelihood Estimator (MLE) asymptotically for $K \gg 1$

$$\boldsymbol{\xi}_{\boldsymbol{\mu}}^{\mathrm{ML}} \equiv \arg\max_{\boldsymbol{\phi}}\, p(\boldsymbol{\mu}\,|\,\phi)$$

Matteo G. A. Paris, Int. J. Quant. Inf. 7, 125 (2009)
Jing Liu et al 2020 J. Phys. A: Math. Theor. 53 023001

Cramer-Rao bound and Fisher information approach

- We impose unbiasedness constraint on estimators: $\langle \boldsymbol{\xi}_m \rangle_{p_{\boldsymbol{\phi}}(m)} = \boldsymbol{\phi}$
- Then the MSE for K measurements is lower bounded by the chain of inequalities

$$K \times \mathrm{MSE}(\boldsymbol{\phi}) \geq \Delta_{\mathrm{CRB}} \geq \Delta_{\mathrm{QCRB}} \geq \Delta_{\mathrm{HL}} \sim \frac{d^2}{N^2}$$

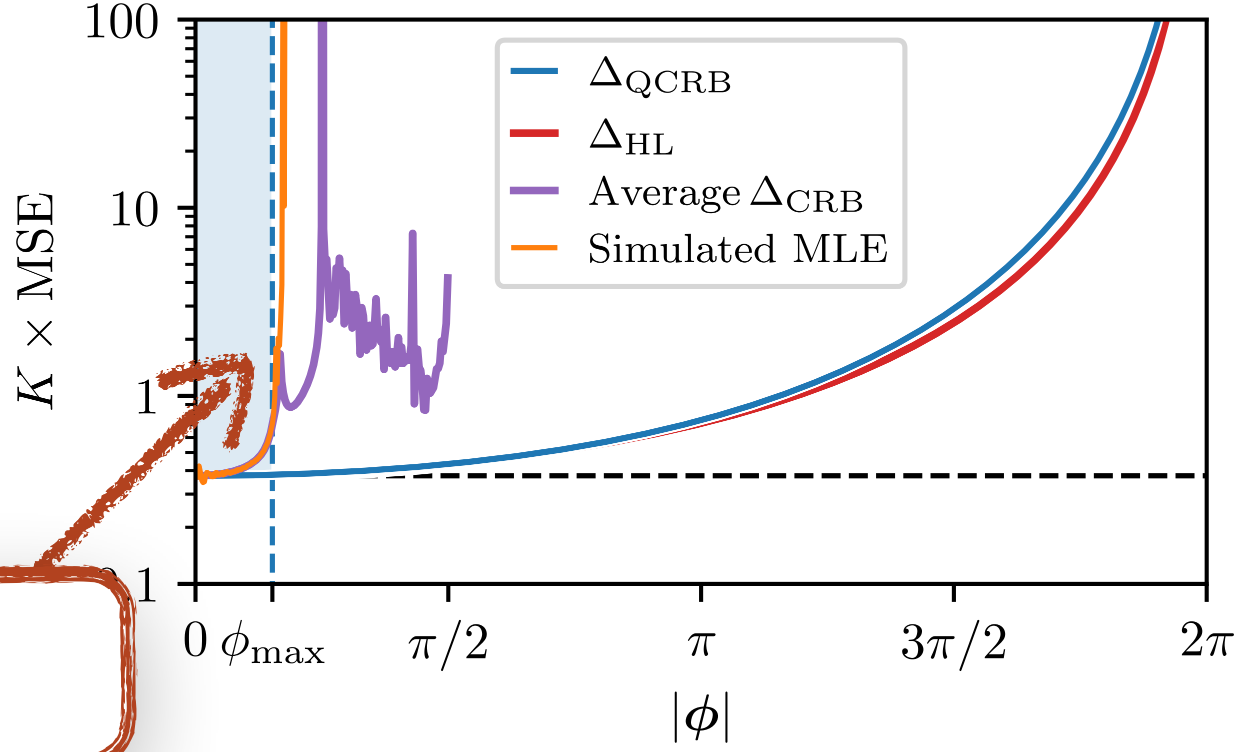
optimised over estimators and measurements and state

- The CRB is saturated by the Maximum Likelihood Estimator (MLE) for $K \gg 1$

$$\boldsymbol{\xi}^{\mathrm{ML}}_{\boldsymbol{\mu}} \equiv \arg\max_{\boldsymbol{\phi}} p(\boldsymbol{\mu} \,|\, \phi)$$

it is *unambiguous* only within a certain domain in parameter space.

Example:
3D sensing, $\boldsymbol{\phi} = \{\phi_x, \phi_y, \phi_z\}$
with 4 qubits



How to find the optimal sensor?

Matteo G. A. Paris,
Jing Liu et al 2020

Optimality Criteria

- Our goal is to formulate a cost function which identifies the optimal sensor with the following properties:

(i) Achieves the QCRB (or HL) at a given phase value of interest $\boldsymbol{\phi}_0$

(ii) Maximizes the domain of unambiguous estimation around $\boldsymbol{\phi}_0$

- Straightforward solution:

$$\Delta_{\mathrm{CRB}} = \mathrm{Tr}\{F^{-1}\}, \qquad \text{Fisher matrix } F = \sum_{\mu} \frac{\nabla p(\mu|\phi)\nabla^T p(\mu|\phi)}{p(\mu|\phi)}$$

Minimize the CRB over input states $|\psi_{\mathrm{in}}\rangle$ and measurements M_μ and use the MLE.

- Does not work because the Fisher matrix is agnostic to criterion (ii)

S. Kurdziałek et al, PRL 130, 160802 (2023)

Issue of Estimation Unambiguity

Estimation (Un)Ambiguity

- Example: Estimation of a single phase ϕ in the vicinity of $\phi_0 = 0$

  Optimal sensor is the GHZ state-based interferometer

- Optimal input state

  $$|\,\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|\downarrow\rangle^{\otimes N} + |\uparrow\rangle^{\otimes N}),$$

  the quantum Fisher $F_Q = N^2$

- Optimal measurement basis

  $$|\pm\rangle = \frac{1}{\sqrt{2}}(e^{i\theta}|\downarrow\rangle^{\otimes N} \pm ie^{-i\theta}|\uparrow\rangle^{\otimes N}),$$

  the classical Fisher $F = F_Q = N^2$

  is *independent* of θ

Estimation (Un)Ambiguity

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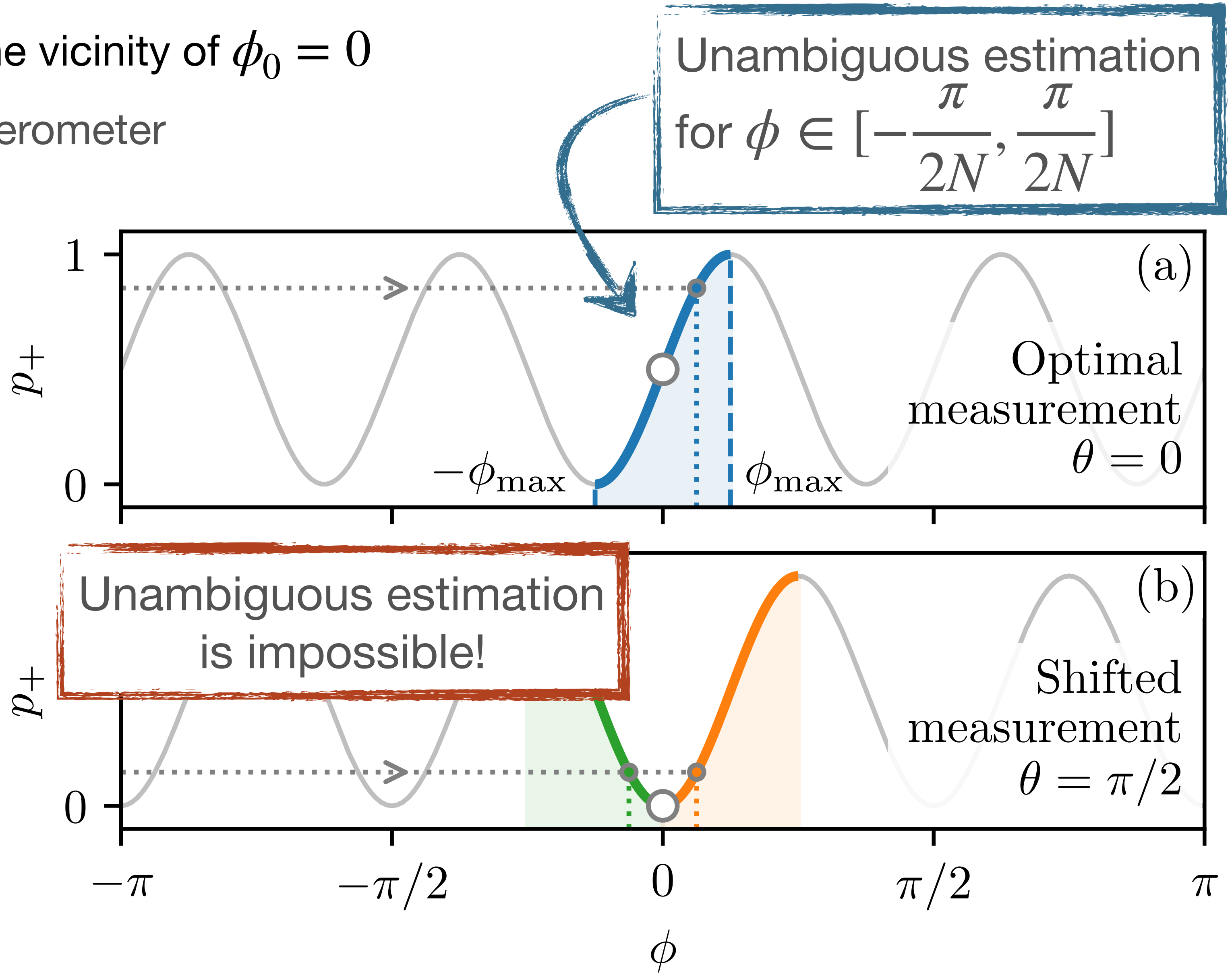
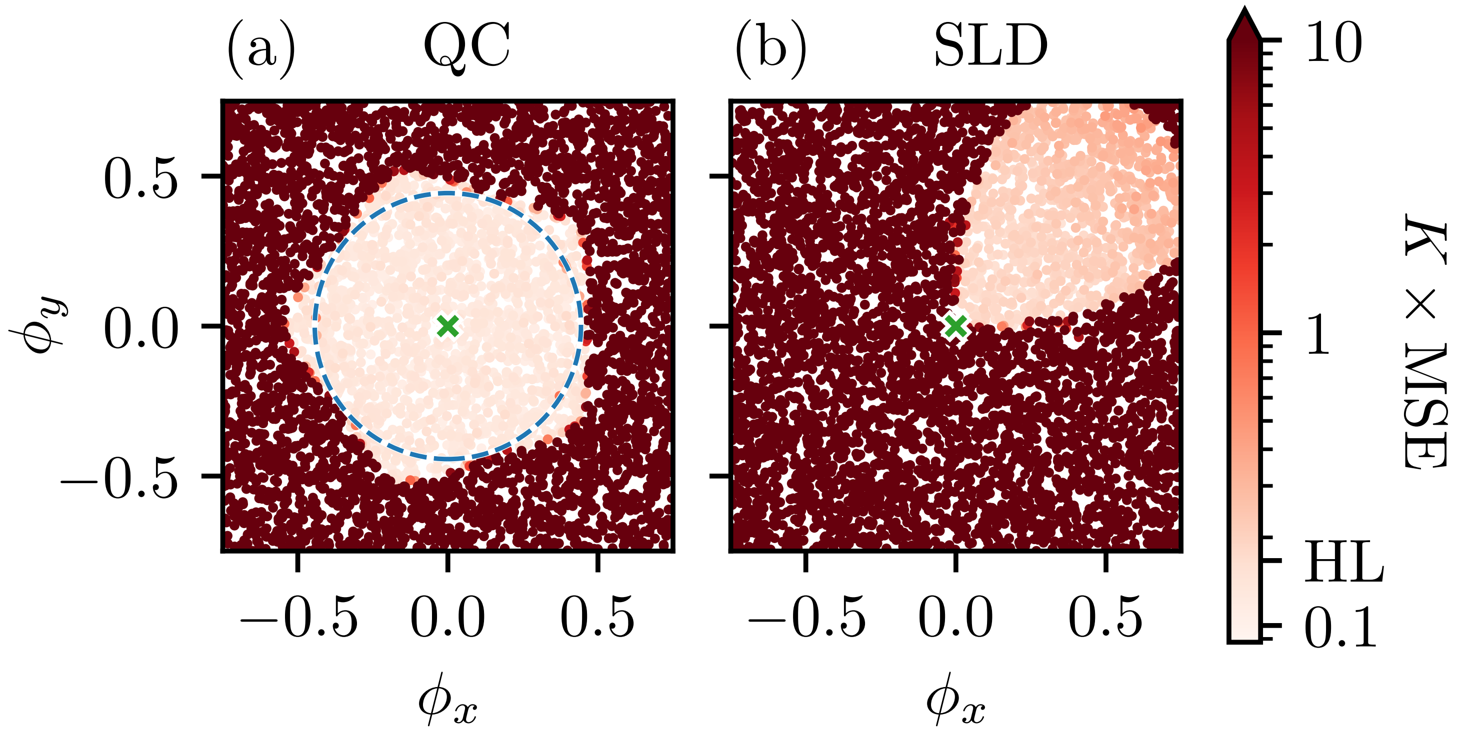


2D Example of Estimation (Un)Ambiguity

2D vector field sensing

$\boldsymbol{\phi} = \{\phi_x, \phi_y\}$

$U = e^{-i\boldsymbol{\phi}\cdot\boldsymbol{J}}$



Simulation of estimation using $K = 10^5$ measurements and MLE.

$N = 4$ atoms

2D Quantum Compass

Symmetric Logarithmic Derivative (SLD) based measurement.

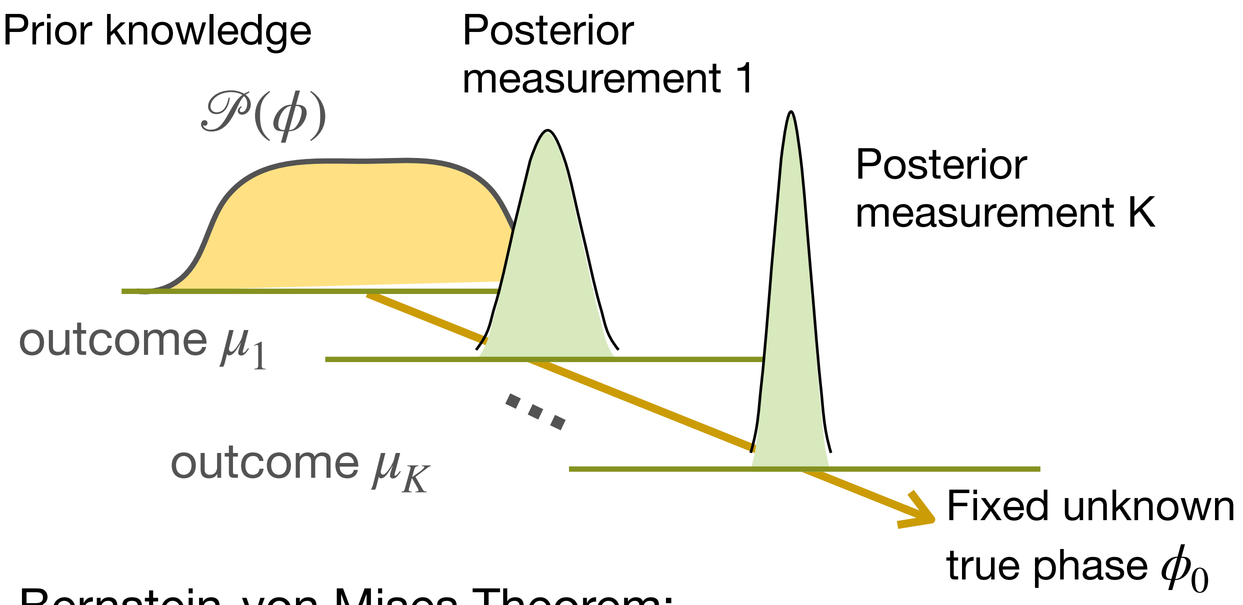
It involves a projector onto the evolved state $|\psi_{\boldsymbol{\phi}_0}\rangle\langle\psi_{\boldsymbol{\phi}_0}|$ and projectors defined by the SLDs

T. Baumgratz and A. Datta, PRL 116, 030801 (2016).

Asymptotic Bayesian Cost
as
Metrological Cost Function

Metrological Cost Function: Asymptotic Bayesian Cost

K measurement repetitions

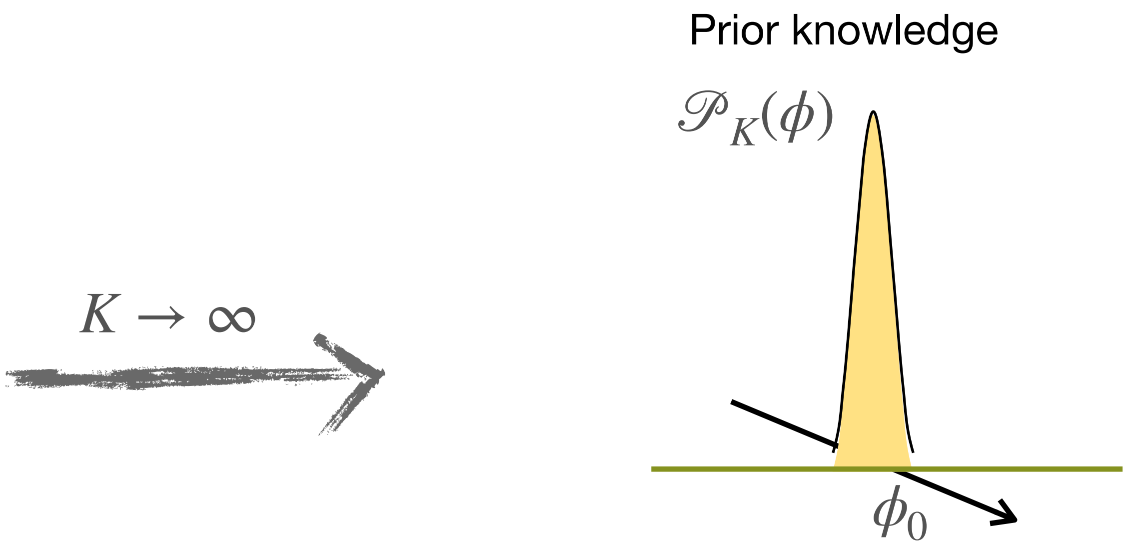


Bernstein-von Mises Theorem:

$$p(\boldsymbol{\phi}\,|\,\boldsymbol{\mu}) \to \mathscr{P}_K(\phi) \propto \exp\left[-\frac{K}{2}(\phi-\phi_0)F(\phi-\phi_0)\right],$$

posterior for $K \gg 1$ — Fisher Information matrix

Single shot Bayesian cost



$$\Xi_K = \sum_\nu \int d\boldsymbol{\phi}\,(\boldsymbol{\phi}-\zeta_\nu)^2 p(\nu\,|\,\boldsymbol{\phi})\mathscr{P}_K(\boldsymbol{\phi}-\boldsymbol{\phi}_0)$$

Kaubruegger, et.al., PRX QUANTUM **4,** 020333 (2023)

Metrological Cost Function: Asymptotic Bayesian Cost

- Ξ_K can be efficiently optimized numerically with respect to the state, measurement, and estimator

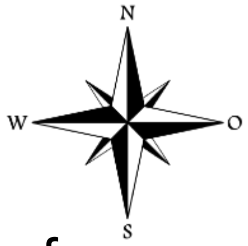
$$|\psi_{\text{in}}\rangle_K,\ \{M_\mu\}_K = \arg\min_{|\psi_{\text{in}}\rangle, \{M_\mu\}, \zeta_\mu} \Xi_K$$

- State and measurement in the limit $K \to \infty$

$$|\psi_{\text{in}}{}^*\rangle,\ \{M_\mu^*\}_K = \lim_{K\to\infty} \left[|\psi_{\text{in}}\rangle_K,\ \{M_\mu\}_K \right]$$

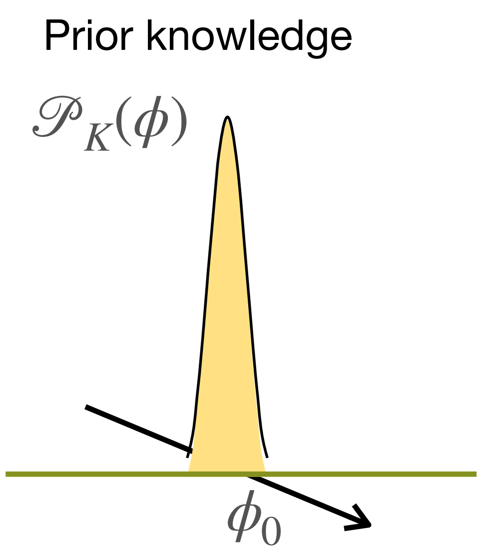
fulfill criteria (i) and (ii)

- We refer to the limiting solutions as *Quantum Compass Solutions*



- QCs define the limiting performance for multiparameter estimation, see arXiv:2404.14194

Single shot Bayesian cost



$$\Xi_K = \sum_\nu \int d\boldsymbol{\phi}\, (\boldsymbol{\phi} - \boldsymbol{\zeta}_\nu)^2 p(\nu | \boldsymbol{\phi}) \mathscr{P}_K(\boldsymbol{\phi} - \boldsymbol{\phi}_0)$$

R. Kaubruegger, A. Shankar, DV, P. Zoller, PRX QUANTUM **4**, 020333 (2023)

Quantum Compass Solutions

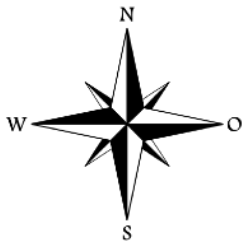
Optimal two parameter estimation with SU(2) sensor

Initial state:

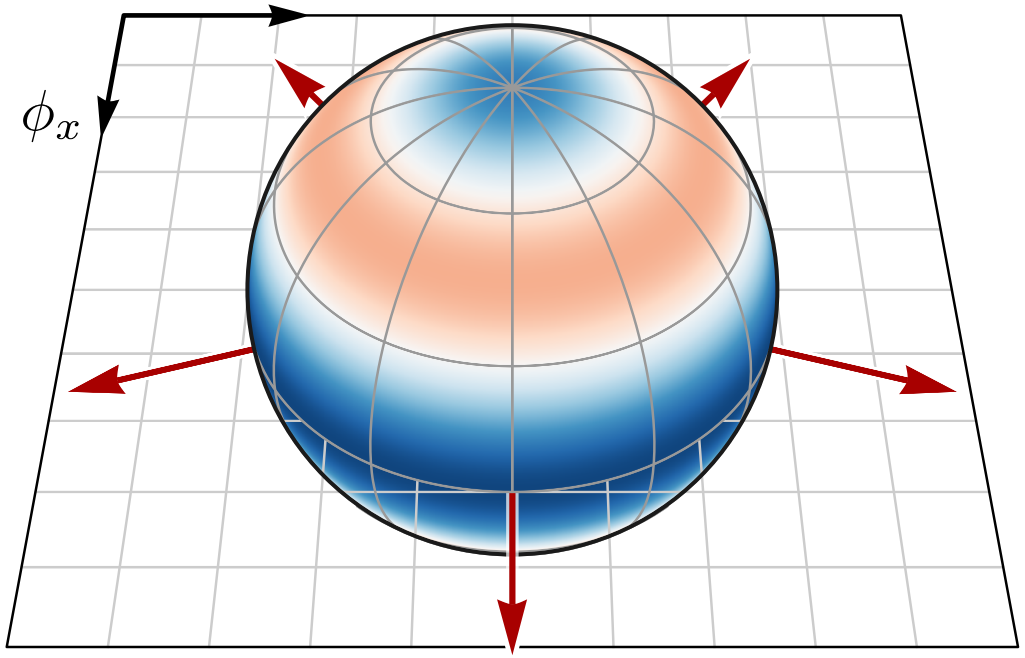
$$|\psi_{\text{in}}\rangle = |J = \tfrac{N}{2}, m = 0\rangle$$

Measurement basis:

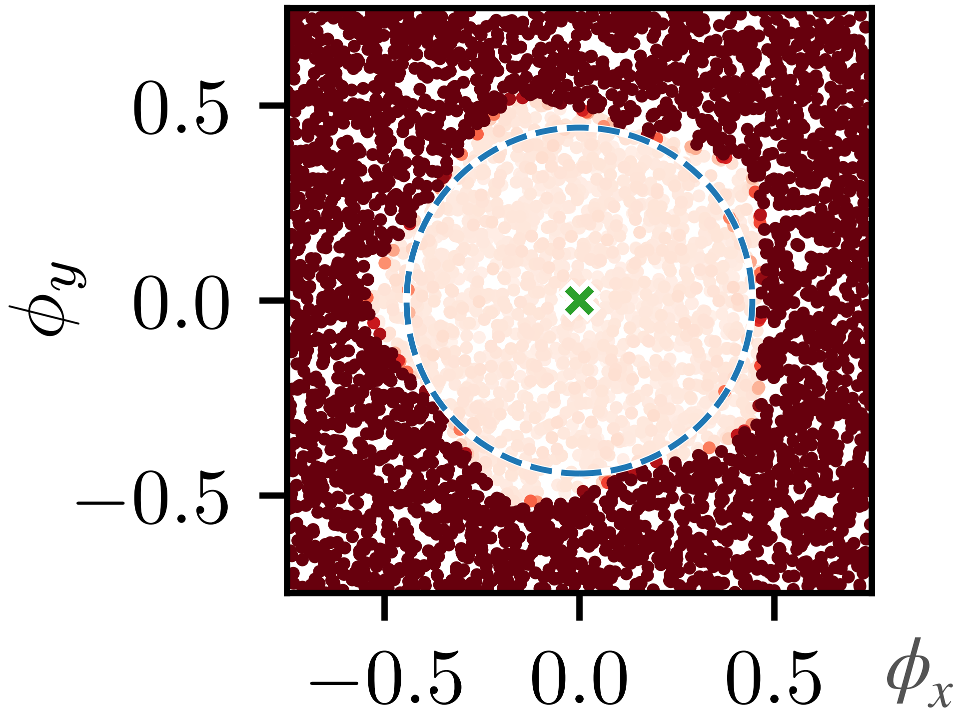
$$|\mu\rangle = \frac{1}{\sqrt{2J+1}} \sum_{m=-N/2}^{N/2} e^{-i\frac{\pi}{2}\left(\frac{4\mu m}{2J+1} - |m|\right)} |\tfrac{N}{2}, m\rangle$$



Wigner function of $|\psi_{\text{in}}\rangle$ and
Compass-like single-shot estimators $\boldsymbol{\xi}_\mu$



MLE performance of
the 2D QC solution, $N = 4$

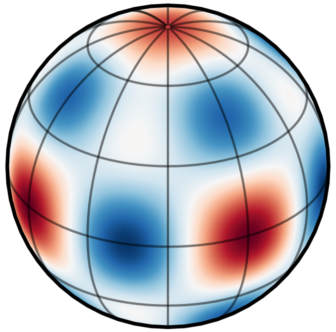


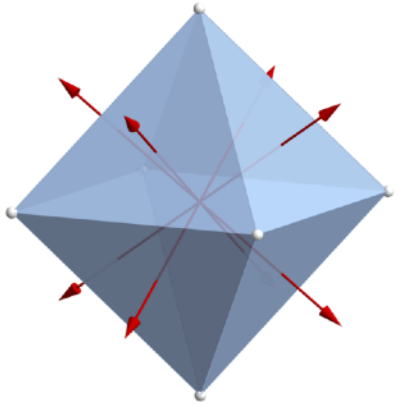
Optimal three parameter estimation with SU(2) sensor

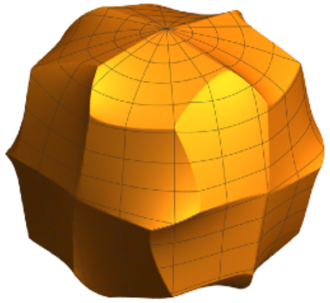
Wigner $|\Psi_{in}\rangle$ Majorana $|\Psi_{in}\rangle$ Domain Ω^*

$N = 6$

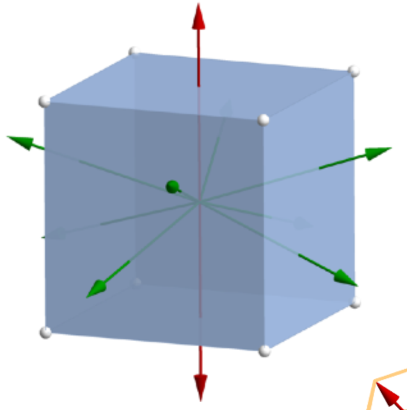
$$\Psi_{\pm 1} = \frac{1}{\sqrt{2}}$$

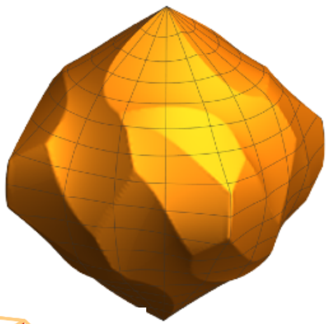




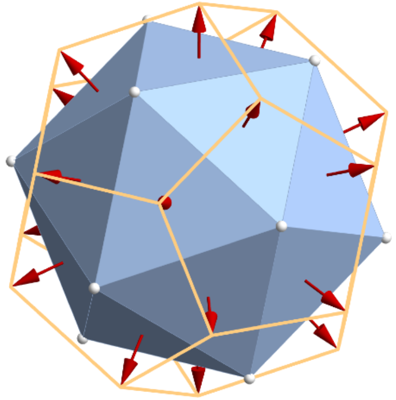


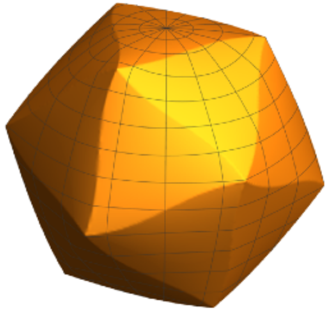
$N = 8$





$N = 12$





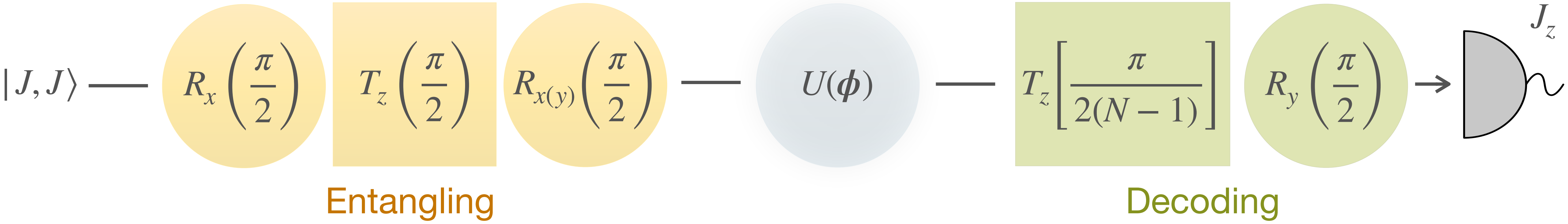
QC is a direct multiparameter counterpart to the single-parameter GHZ-state based interferometer

Approaching the Optimal Sensing Performance
Using
Variational Quantum Circuits

The Simplest Circuit: Optimal Detection of the GHZ state

Phases are encoded via $U(\boldsymbol{\phi}) = \exp[-i(\phi_x J_x + \phi_y J_y + \phi_z J_z)]$.

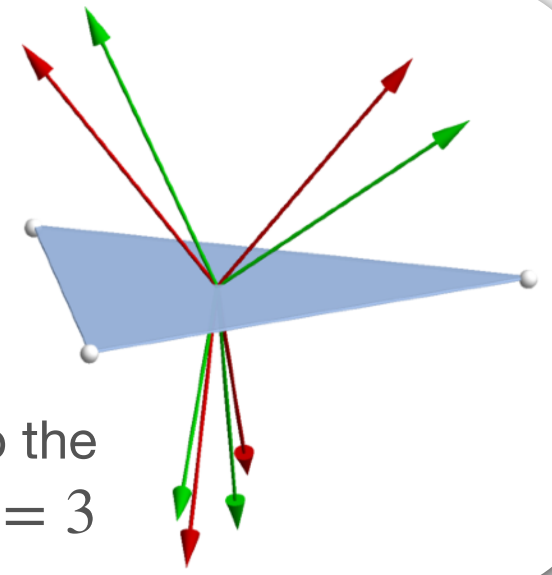
We use global rotations $R_\alpha(\theta) = e^{-i\theta J_\alpha}$ and one-axis twisting $T_\alpha(\theta) = e^{-i\theta J_\alpha^2}$



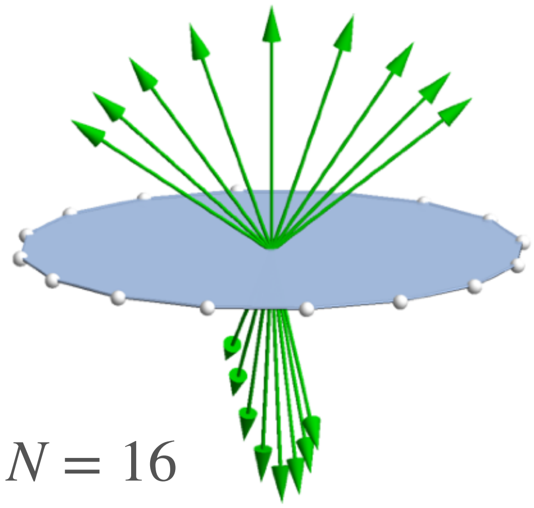
Fisher matrix is anisotropic:

$$F = \begin{pmatrix} N & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & N^2 \end{pmatrix}$$

The HL can be reached by alternating state and measurement orientations.



Close to the QC for $N = 3$



$N = 16$

Next lecture

1. Beyond Spin Squeezing: Variational Quantum Metrology for Ramsey Interferometry and Atomic Clocks

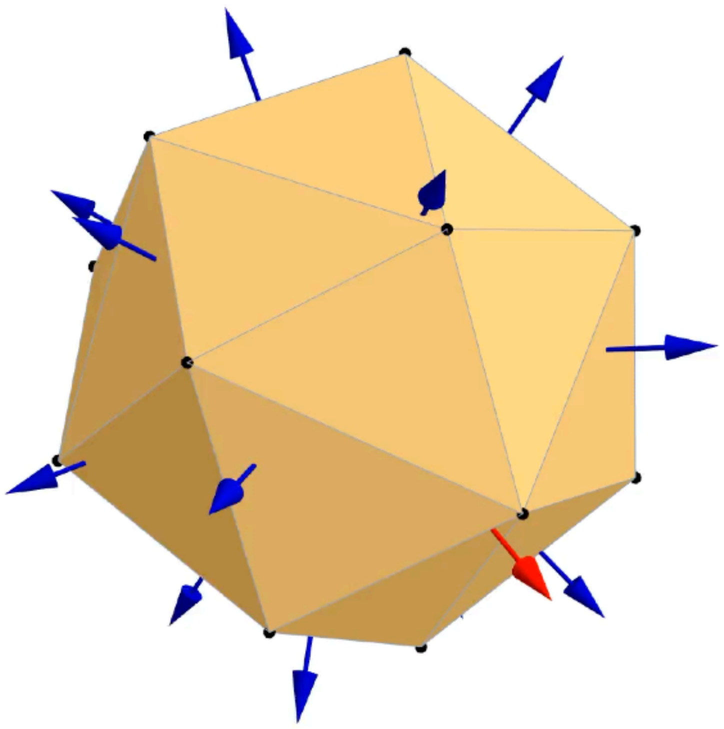
2. Programmable Quantum Sensors and Quantum Compasses

3. Quantum Sensing Networks for Tests of Quantum Mechanics and General Relativity

Open positions:
- PhD students
- Postdocs

Conclusion

- Systematic method to identify the optimal quantum sensor, achieving fundamental precision bounds in the general setting of multiparameter estimation.
- Analytical QC solutions for 2- and 3-parameters estimation using SU(2) sensors.
- Identify higher-order information matrices as a crucial ingredient for defining optimal sensors.
- Open avenues for quantum variational techniques to design low-depth quantum circuits approaching the optimal sensing performance in the many-repetition scenario.



arXiv:2404.14194